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EVALUATION SUBGROUPS OF GENERALIZED HOMOTOPY GROUPS

by

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THIS DISSERTATION IS DEDICATED TO MY FATHER

ABSTRACT

In a series of papers, D.H. Gottlieb studied the evaluation subgroups $G_n(X)$ of homotopy groups extensively. Later, K. Varadarajan generalized $G_n(X)$ to the more general setting $G(A,X)$ and dualized. The purpose of this work is to carry out a further study of the evaluation subgroup and its dual in Varadarajan's setting. In Chapter I we show that cyclicity of maps is closed under product and that if f is cyclic then Ωf is central. The relationship between cyclicity of maps and maps of finite order is also investigated. Chapter II is devoted to the study of $G(A,X)$. Some results of Gottlieb ([9] and [12]) are generalized. A convenient subset $C(A,X)$ of $[A,X]$ (when A is a co-H-space) is introduced and some of its basic properties derived. It is also shown that $G(A,X)$ and $C(A,X)$ are contravariant functors of A from the full subcategory of H-cogroups and maps into the category of abelian groups and homomorphisms. We deduce from this that $G(X,X)$ and $C(X,X)$ are rings if X is an H-cogroup. Chapter III deals with the dual concept cocyclicity of maps. The dual notion of centrality is also introduced and some basic results established. In Chapter IV we settle a problem of Varadarajan by showing that $DG(X,A)$ is a subgroup contained in the center of $[X,A]$ if A is an H-group. Most of the results of Chapter II are dualized by using some facts of Chapter III.

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PRELIMINARIES

This work is a continuation of the study of the evaluation subgroups initiated by B.J. Jiang and D.H. Gottlieb. It was Jiang [22] who first investigated the subgroup $\omega_{\#}(\pi_1(X^X,k))$ of the fundamental group in the study of the Nielsen-Wecken theory of fixed point classes. Later, Gottlieb, in a series of papers ([9], [10], [11], [12], [13] and [14]), studied the evaluation subgroups $G(X)$ (which is $\omega_{\#}(\pi_1(X^X,1_X))$) and $G_n(X)$ (which is $\omega_{\#}(\pi_n(X^X,1_X))$) extensively. More precisely, he studied the connections among the evaluation map and H-spaces, Whitehead products, the Euler characteristic, covering spaces, fibrations, and homology. Subsequent to the work of Gottlieb, contributions have been made in this area by W.J. Barnier [4], T. Ganea [8], H.B. Haslam [15] and [16], C.S. Hoo [20], G.E. Lang, Jr. [23], J. Siegel [29], and K. Varadarajan [31]. Barnier studied the Jiang subgroup in great detail and generalized some results of Gottlieb [9]. Ganea provided an example of a space X in which $P(X) \neq G(X)$. Haslam considered H-spaces mod F, G-spaces mod F and the dual of $G_n(X)$ in Gottlieb's setting. Hoo gave a criterion for cyclic maps from suspensions to suspensions. In his dissertation, Lang obtained a long exact sequence which generalizes the EHP sequence of G.W. Whitehead. Techniques were developed for determining $G(X)$ and calculations of $G_n(X)$ for certain spaces X were also given. Siegel produced a finite dimensional G-space (that is, a space X in which $G_n(X) = \pi_n(X)$

for all n) which is not an H-space. It was Varadarajan who first generalized $G_n(X)$ to $G(A,X)$ and dualized. Some parallel results of and others dual to $G_n(X)$ were obtained in this general setting.

It is our purpose in this work to make a further study of the evaluation subgroups and their duals in Varadarajan's general setting. In Chapter I we consider cyclic maps and explore some of their properties. It is shown that cyclicity of maps is closed under product and that if f is cyclic then Ωf is central. The relationship between cyclicity of maps and maps of finite order is also investigated. Chapter II is devoted to the study of $G(A,X)$ which consists of all homotopy classes of cyclic maps from A to X. Its intrinsic structures are further examined. Some results of Gottlieb ([9] and [12]) are generalized. A larger subset $C(A,X)$ of $[A,X]$ (when A is a co-H-space) is introduced and some of its basic properties derived. It is also shown that $G(A,X)$ and $C(A,X)$ are contravariant functors of A from the full subcategory of H-cogroups and maps into the category of abelian groups and homomorphisms. From this we deduce that $G(X,X)$ and $C(X,X)$ are rings if X is an H-cogroup. Chapter III deals with the dual concept cocyclicity of maps. The dual notion of centrality is also introduced and some basic results established. In Chapter IV we settle a problem of Varadarajan by showing that $DG(X,A)$ is a subgroup contained in the center of $[X,A]$ if A is an H-group. Most of the results of Chapter II are dualized by using some of the results of Chapter III.

We shall now establish the notation and terminology that will be used throughout this thesis. Unless otherwise stated, we shall work in the category of spaces with base points and having the homotopy type of locally finite CW-complexes (which will be defined later). All maps shall mean continuous functions. All homotopies and maps are to respect base points. The base point as well as the constant map will be denoted by $*$. 1 (sometimes with decoration) will denote the identity function (resp. map) of a set or a group (resp. space) when it is clear from the context. For simplicity, we shall use the same symbol for a map and its homotopy class.

In what follows, let A, B, X, Y and Z be any spaces. The set of homotopy classes of maps from X into Y will be denoted by $[X,Y]$. For each map $f\colon X \to Y$, the induced functions

$$f_{\#}\colon [Z,X] \to [Z,Y]$$

and

$$f^{\#}\colon [Y,Z] \to [X,Z]$$

are respectively given by $f_{\#}(g) = fg$ for each $g \in [Z,X]$ and $f^{\#}(h) = hf$ for each $h \in [Y,Z]$.

All function spaces will be endowed with the compact-open topology and, unless otherwise stated, the constant map will be taken to be the base point. X^X shall denote the space of free maps from X to X with 1_X as base point. The <u>evaluation map</u> $\omega\colon X^X \to X$ is defined to be $\omega(f) = f(*)$ for each $f \in X^X$.

The wedge product of X and Y is given by $X \vee Y \equiv X \times \{*\} \cup \{*\} \times Y$, the smash product by

$$X \wedge Y \equiv \frac{X \times Y}{X \vee Y}\,,$$ and the flat product

by $X \flat Y \equiv E(X \times Y; X \vee Y, *)$ which is the space of paths in $X \times Y$ which begin in $X \vee Y$ and end in $*$.

The diagonal map $\Delta\colon X \to X \times X$ is given by $\Delta(x) = (x,x)$ for each $x \in X$, the folding map $\nabla\colon X \vee X \to X$ by $\nabla(x,*) = \nabla(*,x) = x$ for each $x \in X$, and the switching map $T\colon X \times Y \to Y \times X$ by $T(x,y) = (y,x)$ for each $x \in X$, $y \in Y$.

Frequently (not always) i and j will be reserved for the inclusion maps of the form $i_1\colon X \to X \times Y$ or $i_2\colon Y \to X \times Y$, and $j\colon X \vee Y \to X \times Y$ respectively. The projection is denoted by p with decoration.

An H-space X is a topological space together with a map $m\colon X \times X \to X$ such that $mj \simeq \nabla$ where $\simeq$ denotes "homotopic". The map m is called an H-structure (or multiplication) on X.

A co-H-space X is a topological space together with a map $\phi\colon X \to X \vee X$ such that $j\phi \simeq \Delta$. The map ϕ is called a co-H-structure (or comultiplication) on X.

An H-group X is a topological space together with an H-structure $m : X \times X \to X$ and a map $\mu : X \to X$ such that the following are satisfied:

(1) $m(m\times 1) \simeq m(1\times m)$ (homotopy associativity),

(2) $m(1\times *)\Delta \simeq m(*\times 1)\Delta \simeq 1$ (existence of identity), and

(3) $m(1\times\mu)\Delta \simeq m(\mu\times 1)\Delta \simeq *$ (existence of homotopy inverse).

An H-cogroup X is a topological space together with a co-H-structure $\phi : X \to X \vee X$ and a map $\nu : X \to X$ such that the following are satisfied:

(1) $(1\vee\phi)\phi \simeq (\phi\vee 1)\phi$ (homotopy associativity),

(2) $\nabla(1\vee *)\phi \simeq \nabla(*\vee 1)\phi \simeq 1$ (existence of identity), and

(3) $\nabla(1\vee\nu)\phi \simeq \nabla(\nu\vee 1)\phi \simeq *$ (existence of homotopy inverse).

Let I be the closed unit interval. The reduced suspension of X is defined to be

$$\Sigma X \equiv \frac{X \times I}{X\times\{0,1\} \cup \{*\}\times I}$$

and the loop space of X to be $\Omega X \equiv \{\ell \in X^I \mid \ell(0) = \ell(1) = *\}$. If $f: X \to Y$ is a map, then the maps $\Sigma f: \Sigma X \to \Sigma Y$ and $\Omega f: \Omega X \to \Omega Y$ are respectively given by $\Sigma f(x,t) = (f(x),t)$ for each $x \in X$, $t \in I$ and $\Omega f(\ell)(t) = f(\ell(t))$ for each $\ell \in \Omega X$, $t \in I$. The adjoint functor (or natural isomorphism) from the group $[\Sigma X,Y]$ to the group $[X,\Omega Y]$ will be denoted by τ. The symbols e_A and e'_A shall denote $\tau^{-1}(1_{\Omega A})$ and $\tau(1_{\Sigma A})$ respectively, the subscript will be dropped if there is no danger of confusion.

For each $f \in [\Sigma A,X]$ and $g \in [\Sigma B,X]$, there is associated a unique homotopy class in $[\Sigma(A\wedge B),X]$ which is called the generalized Whitehead product of f and g and is denoted by $[f,g]$ (see [1]). Dually, for each $f \in [X,\Omega A]$ and $g \in [X,\Omega B]$, there is associated a unique homotopy class in

$[X,\Omega(A\flat B)]$ which is called the dual of the GWP of f and g and is denoted by $[f,g]'$ (see [1] also).

Let $n > 1$ be an integer. A space X is said to be (n-1)-connected iff $\pi_k(X) = o$ for all $k \leq n-1$.

Let Q be the field of rationals. A space X is said to be a rational homology n-sphere iff

$$\tilde{H}_q(X;Q) = \begin{cases} Q, & \text{for } q = n \\ o, & \text{for } q \neq n . \end{cases}$$

The Euler characteristic of a space X is defined to be

$$\chi(X) = \sum_{n=o}^{\infty} (-1)^n \alpha_n$$

where

$$\alpha_n = \dim(H^n(X;Q)) \quad .$$

Let E^n and e^n denote respectively the closed and the open n-cells $(n \geq 0)$. A CW-complex X is a Hausdorff space, together with an indexing set A_n for each integer $n \geq 0$, and maps $\phi^n_\alpha\colon E^n \to X$ (all $n \geq 0$, $\alpha \in A_n$), such that the following properties are satisfied (see [26]):

(1) $X = \cup\, \phi^n_\alpha(e^n)$, for all $n \geq 0$ and $\alpha \in A_n$ (we interpret e^o and E^o as a single point).

(2) $\phi^n_\alpha(e^n) \cap \phi^m_\beta(e^m)$ is empty unless $n = m$ and $\alpha = \beta$; and $\phi^n_\alpha | e^n$ is one-to-one for all $n \geq 0$ and $\alpha \in A_n$.

(3) Let $X^n = \cup\, \phi^m_\alpha(e^m)$, for all $0 \leq m \leq n$ and all $\alpha \in A_m$. Then $\phi_\alpha(S^{n-1}) \subset X^{n-1}$ for each $n \geq 1$ and $\alpha \in A_n$.

(4) A subset Y of X is closed iff $(\phi^n_\alpha)^{-1}(Y)$ is closed in E^n, for each $n \geq 0$ and $\alpha \in A_n$.

(5) For each $n \geq 0$ and $\alpha \in A_n$, $\phi^n_\alpha(E^n)$ is contained in the union of a finite number of sets of the form $\phi^m_\beta(e^m)$.

A CW-complex X is locally finite iff each point $x \in X$ has a neighborhood meeting only a finite number of cells.

A map $f: X \to Y$ is called a fibration iff for any space Z, any map $g: Z \to X$ and any homotopy $H: Z \times I \to Y$ such that $H(z,o) = fg(z)$, there exists a homotopy $G: Z \times I \to X$ with $G(z,o) = g(z)$ and $fG = H$.

A map $f: X \to Y$ is called a cofibration iff for any space Z, any map $g: Y \to Z$ and any homotopy $H: X \times I \to Z$ such that $H(x,o) = gf(x)$, there exists a homotopy $G: Y \times I \to Z$ with $G(y,o) = g(y)$ and $G(f(x),t) = H(x,t)$ for all $x \in X$, $t \in I$.

For the notation and terminology not mentioned above the reader is referred to [26] or [30] unless otherwise specified.

The following facts are frequently used:

(1) If A is a co-H-space, then we can find a map $s: A \to \Sigma\Omega A$ such that $es \simeq 1_A$.

(2) If B is an H-space, then we can find a map $s': \Omega\Sigma B \to B$ such that $s'e' \simeq 1_B$.

(3) Let A and B be an H-cogroup and an H-group respectively. Then $[A,X]$ and $[X,B]$ are groups for any space X.

CHAPTER I

CYCLIC MAPS

1.1 Introduction

In this chapter we make a further study of cyclic maps. Section 1.2 is devoted to the definition and existence of cyclic maps. In Section 1.3 we show that cyclicity of maps is closed under product and that if f is cyclic then Ωf is central. Some results of Gottlieb on homology together with a result in Section 2.2 of Chapter II, namely $\omega_{\#}([A,X^X]) = G(A,X)$, are applied in Section 1.4 to investigate the relationship between cyclicity of maps and maps of finite order.

1.2 Definition and Existence of Cyclic Maps

Definition 1.2.1. A map $f: A \to X$ is said to be cyclic if there exists a map $F: X \times A \to X$ such that the following diagram is

$$\begin{tikzcd} X \times A \arrow[r, "F"] & X \\ X \vee A \arrow[u, "j"] \arrow[ur, "\nabla(1\vee f)"'] \arrow[u, phantom, shift left=4, "\simeq"] & \end{tikzcd}$$

homotopy commutative: that is, $Fj \simeq \nabla(1\vee f)$. Since j is a cofibration, this is equivalent to saying that we can find a map G $G: X \times A \to X$ such that $Gj = \nabla(1\vee f)$. We call such a map G an associated map of f. The set of all homotopy classes of cyclic maps from A to X is denoted by $G(A,X)$ and is called the Gottlieb subset of $[A,X]$.

Remark 1. Note that a map $f: A \to X$ is cyclic iff there exists a map $F: X \times A \to X$ such that $Fi_1 = 1_X$ and $Fi_2 = f$, where

$i_1: X \to X \times A$ and $i_2: A \to X \times A$ are inclusions, that is, F is of type $(1,f)$. Clearly the constant function $*: A \to X$ is cyclic.

Remark 2. If $A = S^n$ ($n \geq 1$ is an integer), then $G(A,X)$ reduces to $G(X)$ ([9]) and $G_n(X)$ ([12]) which is called the n^{th} evaluation subgroup of X.

Lemma 1.2.2 ([31]). Let $f: A \to X$ be a cyclic map and $\theta: B \to A$ an arbitrary map. Then $f\theta: B \to X$ is a cyclic map.

The existence of cyclic maps is easily seen from the previous remark and the following proposition.

Proposition 1.2.3. Let X be a space. Then the following are equivalent:

(a) X is an H-space.

(b) 1_X is cyclic.

(c) $G(A,X) = [A,X]$ for any space A.

Proof. (a) => (b). Let m be the H-structure on X. Then $mj \simeq \nabla = \nabla(1_X \vee 1_X)$, so that 1_X is cyclic.

(b) => (c). Let A be any space and let $f \in [A,X]$. Then $f = 1_X \circ f$ is cyclic, by Lemma 1.2.2.

(c) => (a). Take $A = X$. Then 1_X is cyclic, so that we can find a map $m: X \times X \to X$ such that $mj = \nabla$.

Thus we have the following result of Varadarajan and also its converse:

Corollary 1.2.4 ([31]). If X is an H-space, then any map $f: A \to X$ is cyclic.

Another way in which cyclic maps arise naturally is by fibrations. Suppose $F \to E \to B$ is a fibration. Then we have an operation $\rho\colon F \times \Omega B \to F$ of the loop space of the base on the fibre. The fibration gives rise to a Puppe sequence $\ldots \to \Omega E \to \Omega B \xrightarrow{\partial} F \to E \to B$. We can take $\partial = \rho|\Omega B$, that is, ρ is a map of type $(1,\partial)$, or ∂ is cyclic. It follows that for all spaces A, $\partial_{\#}[A,\Omega B] \subset G(A,F)$. If $G(A,F) = o$, then we can obtain some information on the fibration. More precisely, we have the following result.

Theorem 1.2.5. Let $F \to E \xrightarrow{p} \Sigma A$ be a fibration. If $G(A,F) = o$, then p has a cross-section.

Proof. If $G(A,F) = o$, then $\partial_{\#}[A,\Omega\Sigma A] = o$. Hence from the exact sequence of the fibration, we see that $(\Omega p)_{\#}\colon [A,\Omega E] \to [A,\Omega\Sigma A]$ is onto. In particular, we can find a map $f\colon A \to \Omega E$ such that $(\Omega p)f \simeq e'$ where $e'\colon A \to \Omega\Sigma A$ is the adjoint of the identity map $\Sigma A \to \Sigma A$. Taking adjoints we obtain $p\tau^{-1}(f) \simeq 1_{\Sigma A}$ where $\tau^{-1}(f)$ is the adjoint of f. Thus we obtain a cross-section.

Example. Any fibration $E \to S^{2n+1}$ with fibre S^{2n} $(n \geq 1)$ admits a cross-section. In fact, $G_{2n}(S^{2n}) = o$, by Theorem 5.4 of [12].

A third way of getting cyclic maps is as follows: Let G be a topological group and let H be a closed subgroup. Let G/H denote the space of left cosets and let $p\colon G \to G/H$ be the natural map. Then p is cyclic since we have a natural map $G/H \times G \to G/H$ given by $(g_1H,g_2) \to g_2g_1H$, of type $(1,p)$. For further detail about this see [23] and also Proposition 2.3.9.

1.3 Some Basic Properties of Cyclic Maps

Let $\theta\colon B \to A$ and $g\colon X \to Y$ be maps such that g has a right homotopy inverse. Then if $f\colon A \to X$ is cyclic, so are $f\theta$ and gf.

Example 1. Let A be a co-H-space and $f\colon A \to X$ a map. Then f is cyclic iff $fe\colon \Sigma\Omega A \to X$ is cyclic. In fact, there exists a map $s\colon A \to \Sigma\Omega A$ such that $es \simeq 1_A$, so that $f \simeq fes$.

Example 2. Let $\alpha \in \pi_{2n+1}(X)$ be such that any representative g of α has a right homotopy inverse, then $2\alpha \in G_{2n+1}(X)$. For if $f\colon S^{2n+1} \to S^{2n+1}$ is a map of degree 2, then $2[g] = [gf]$ and f is cyclic, by Theorem 5.4 of [12]. Thus gf is cyclic.

Definition 1.3.1 ([21]). A space X is said to be *m-coconnected* ($m \geq 1$ is an integer) if $H^q(X;G) = o$ for each $q \geq m$ and for each coefficient group G.

Lemma 1.3.2 ([21], p. 213). Let X and Y be two $(2m-1)$-coconnected spaces. If $f\colon X \to Y$ is a map, then $f^{\#}\colon \pi^m(Y) \to \pi^m(X)$ is a homomorphism, where $\pi^m(X)$ and $\pi^m(Y)$ are the m^{th} cohomotopy groups of X and Y respectively.

Proposition 1.3.3. Let B be $(4n+1)$-coconnected. If $g\colon B \to S^{2n+1}$ is any map, then $2g$ is cyclic.

Proof. If $n = o$, then $[B,S^1] = G(B,S^1)$ since S^1 is an H-space. Thus the proposition is true for $n = o$.

Assume $n > o$. Then S^{2n+1} is $(4n+1)$-coconnected. Let

$f\colon S^{2n+1} \to S^{2n+1}$ be a map of degree 2 and ι a generator of $\pi^{2n+1}(S^{2n+1}) = \pi_{2n+1}(S^{2n+1}) = Z$. According to Lemma 1.3.2, $g^{\#}\colon \pi^{2n+1}(S^{2n+1}) \to \pi^{2n+1}(B)$ is a homomorphism since B and S^{2n+1} are $(4n+1)$-coconnected. Hence $g^{\#}(f) = g^{\#}(2\iota) = g^{\#}(\iota+\iota) = g^{\#}(\iota) + g^{\#}(\iota) = g+g = 2g$, so that $fg = 2g$. Now since f is cyclic, it follows that $2g$ is also cyclic.

The next result says that cyclicity of maps is closed under product.

<u>Proposition 1.3.4</u>. If the maps $f\colon A \to X$ and $g\colon B \to Y$ are cyclic, then so is $f \times g\colon A \times B \to X \times Y$.

<u>Proof</u>. Let F and G be two associated maps of f and g respectively. Let $H = (F\times G)(1\times T\times 1)\colon (X\times Y) \times (A\times B) \to X \times Y$. Then H is an associated map of $f \times g$.

It might be supposed that if $f\colon A \to X$ and $g\colon B \to Y$ are cyclic then so is $f \vee g\colon A \vee B \to X \vee Y$. That this is not true can be illustrated by the following example.

<u>Example</u>. Let $A = B = X = Y = S^1$. Then 1_{S^1} is cyclic. But $1_{S^1} \vee 1_{S^1} = 1_{S^1 \vee S^1}$ is not cyclic by Proposition 1.2.3, for $S^1 \vee S^1$ is not an H-space.

<u>Lemma 1.3.5</u>. If the maps $f\colon A \to X$ and $g\colon B \to X$ are cyclic, then so is $\nabla(f\vee g)\colon A \vee B \to X$.

<u>Proof</u>. Let F and G be two associated maps of f and g respectively. Let $h\colon X \times (A\times B) \to (X\times A) \times B$ be the homeomorphism.

Let $K \equiv G(F\times 1_B)h\colon X \times (A\times B) \to X$ and $H \equiv K|X \times (A\vee B) \to X$. Then H is an associated map of $\nabla(f\vee g)$.

The next corollary is an immediate consequence of Lemmas 1.2.2 and 1.3.5.

Corollary 1.3.6. If A is a co-H-space, then $G(A,X) \subset [A,X]$ is closed under the natural operation induced by the co-H-structure on A.

We record the following definition and results which will be needed in the next chapter.

Definition 1.3.7 ([3]). Let (G,m,μ) be an H-group and A any space. We say that a map $f\colon A \to G$ is central if $c(1\times f) \simeq *$ where $c\colon G \times G \to G$ is the basic commutator map (that is, $c \equiv m(m\times m)(1\times 1\times\mu\times\mu)\Delta)$.

Lemma 1.3.8 ([3]). (a) Let $p_1\colon G \times A \to G$ and $p_2\colon G \times A \to A$ be the projections. Then f is central iff the commutator $(p_1,fp_2) \equiv p_1 + fp_2 - p_1 - fp_2 \simeq *$ in $[G\times A,G]$.

(b) Any central map $f\colon A \to G$ lies in the center of $[A,G]$.

(c) Let $f\colon A \to G$ be central and $\theta\colon B \to A$ an arbitrary map. Then $f\theta\colon B \to G$ is central.

Let $f\colon A \to X$ be a map. It is evident from the preceding lemma that if Ωf is central, then $(\Omega f)_{\#}\colon [Z,\Omega A] \to [Z,\Omega X]$ has image contained in the center of $[Z,\Omega X]$ for all spaces Z.

The following lemma is due to Ganea:

Lemma 1.3.9 ([7]). Let $X \flat A \xrightarrow{L} X \vee A \to X \times A$ be a fibration. Then $\nabla(1 \vee f)L \simeq *$ iff Ωf is central.

Lemma 1.3.10. If f is cyclic, then Ωf is central.

Proof. Since f is cyclic, $\nabla(1 \vee f)$ extends to a map $X \times A \to X$, so that $\nabla(1 \vee f)L \simeq *$ and the assertion follows from the preceding lemma.

Remark. In certain situations, the converse of the above lemma is also true (see, for example, Theorem 2.3.2).

1.4 Cyclicity of Maps and Maps of Finite Order

In this section we make some further observations on cyclic maps using some results of Gottlieb [14] on homology. Following [14], we observe that composition of maps makes X^X an H-space with 1_X as base point, where X^X is the space of free maps from X to X. If μ is the composition map, then we can define a multiplication on $H_*(X^X;Q)$ by $xy = \mu_*(x \otimes y)$ for all x,y in $H_*(X^X;Q)$, where Q is the field of rationals. With the diagonal map $\Delta\colon X^X \to X^X \times X^X$ inducing a co-algebra structure on $H_*(X^X;Q)$ it follows that $H_*(X^X;Q)$ is a Hopf algebra [14]. We say that an element $\lambda \in H_n(X^X;Q)$ is primitive if $\Delta_*(\lambda) = 1 \otimes \lambda + \lambda \otimes 1$. The following result is essentially due to Gottlieb [14].

Lemma 1.4.1. Let $\omega\colon X^X \to X$ be the evaluation map. Suppose that $H_*(X)$ is finitely generated. Let $\lambda \in H_n(X^X;Q)$ be primitive and suppose that $\omega_*(\lambda) \neq o$. Then if the Euler characteristic

$\chi(X) \neq o$, we have $\omega_*(\lambda^k) \neq o$ for all $k > o$.

Now suppose that $f\colon A \to X$ is a map satisfying cat $f < 2$, that is, we can find a map $\phi\colon A \to X \vee X$ such that $j\phi \simeq \Delta f$. It is then easily checked that for all α in $H_*(A;Q)$, $\Delta_* f_*(\alpha) = f_*(\alpha) \otimes 1 + 1 \otimes f_*(\alpha)$. In particular, this means that if f is actually a map $A \to X^X$, we have that $f_*(\alpha)$ is primitive. Such would be the situation if A were a co-H-space, for we might let $\phi = \psi f$ where ψ is the co-H-structure on A.

<u>Theorem 1.4.2</u>. Suppose $f\colon A \to X$ is cyclic where A is a co-H-space. Suppose A is a finite dimensional CW-complex and $H_*(X)$, $\pi_*(X)$ are finitely generated. If $\chi(X) \neq o$, then Σf is an element of finite order in $[\Sigma A, \Sigma X]$.

<u>Proof</u>. Since f is cyclic, we can find a map $g\colon A \to X^X$ such that $\omega g \simeq f$ (see Theorem 2.2.2 in the next chapter). Now since A is a co-H-space, it follows that cat $g < 2$, and hence $g_*(\alpha) \in H_*(X^X;Q)$ is primitive for all α in $H_*(A;Q)$. We claim that $f_* = o\colon \tilde{H}_*(A;Q) \to \tilde{H}_*(X;Q)$. For if not, then we can find an element $\alpha \in H_n(A;Q)$ for some $n > o$ such that $f_*(\alpha) \neq o$. Since $g_*(\alpha)$ is primitive and $\omega_*(g_*(\alpha)) = f_*(\alpha) \neq o$, it follows from Lemma 1.4.1 that $\omega_*(g_*(\alpha)^k) \neq o$ for all $k > o$. According to Theorem 1 of [14], we have $H_*(X;Q) \cong \omega_*[g_*(\alpha)]_\infty \otimes M_\infty$ as vector spaces over Q, where $[g_*(\alpha)]_\infty$ is the subspace of $H_*(X^X;Q)$ generated by 1, $g_*(\alpha)$, $g_*(\alpha)^2, \ldots$ if the dimension of α is even, and generated by 1, $g_*(\alpha)$ if the dimension of α is odd, and M_∞ denotes the elements of $H_*(X;Q)$ of depth zero (see [14]

for details and definitions). If the dimension of α is even, this would contradict the fact that $H_*(X)$ is finitely generated, and if the dimension of α is odd, it would contradict the fact that $\chi(X) \neq o$. Hence $f_* = o$. By duality, we have that $f^* = o: \tilde{H}^*(X;Q) \to \tilde{H}^*(A;Q)$. Now consider the map $h = e'f: A \to X \to \Omega\Sigma X$ where $e': X \to \Omega\Sigma X$ is the usual map. Then $h^* = f^* e'^* = o: \tilde{H}^*(\Omega\Sigma X;Q) \to \tilde{H}^*(A;Q)$. According to [2], it now follows that $h = e'f$ is an element of finite order in $[A,\Omega\Sigma X]$. Hence its adjoint Σf is an element of finite order in $[\Sigma A,\Sigma X]$.

Corollary 1.4.3. Let A be a co-H-space which is a CW-complex of dimension $<2n-1$, and suppose that X is $(n-1)$-connected. Suppose that $H_*(X)$ is finitely generated and $\chi(X) \neq o$. If $f: A \to X$ is cyclic, then f is an element of finite order in $[A,X]$.

Corollary 1.4.4. Suppose that X in $(n-1)$-connected and $H_*(X)$ is finitely generated and $\chi(X) \neq o$. If f is an element of $G_m(X)$, we have that Σf is an element of finite order in $\pi_{m+1}(\Sigma X)$. In particular, if $m < 2n-1$, then f is an element of finite order.

The following result is essentially due to Gottlieb (see Theorem 5 of [14]). We state it here for co-H-spaces in general instead of for suspensions.

Lemma 1.4.5. Suppose X is a co-H-space. If $\omega_*: \tilde{H}_*(X^X;Q) \to \tilde{H}_*(X;Q)$ is non-zero, then X is a rational homology n-sphere for some odd integer n.

Corollary 1.4.6. Let X be a co-H-space which is not a rational homology n-sphere, where n is odd, and let A be a finite dimensional CW-complex. If $f: A \to X$ is cyclic, then Σf is an element of finite order.

Proof. Under the hypotheses, we have $\omega_* = o$. Hence since f is cyclic, we have $f_* = o$. The rest of the proof goes along the same lines as the arguments in the proof of Theorem 1.4.2.

Similarly, we have the following result.

Theorem 1.4.7. Let A be a finite dimensional CW-complex and let X be a space such that ΣX is not a rational homology n-sphere, n odd. Let $f: A \to X$ be a map. If Σf is cyclic, then it is of finite order.

Corollary 1.4.8. Let $f: A \to X$ be a map where X is a homotopy associative H-space and A is a finite dimensional CW-complex. If Σf is cyclic then f is an element of finite order.

Proof. According to the hypotheses, it follows that ΣX is not a rational homology n-sphere, n odd. Since Σf is cyclic, by Theorem 1.4.7, it follows that Σf is an element of finite order. Taking adjoints, we see that $e'f$ is of finite order. Thus there exists a positive integer k such that $k(e'f) = o$. But since X is a homotopy associative H-space, there exists an H-map $s': \Omega\Sigma X \to X$ such that $s'e' \simeq 1_X$. Thus $0 = s'k(e'f) = ks'e'f = kf$ and hence f is of finite order.

<u>Note</u>: The conditions that $H_*(X)$ and $\pi_*(X)$ be finitely generated might imply that X is contractible. This would be the case if X were required to be a finite compact complex.

CHAPTER II

EVALUATION SUBGROUPS OF GENERALIZED HOMOTOPY GROUPS

2.1 Introduction

Recall that $G(A,X)$ consists of all homotopy classes of cyclic maps from A to X. In general, it is not a group but is known to be a subgroup of $[A,X]$ if A is an H-cogroup. A result of basic importance, that is $\omega_{\#}([A,X^X]) = G(A,X)$, is established in Section 2.2. It is also shown that $G(A,X)$ preserves products in the second variable. In Section 2.3, a convenient subset $C(A,X)$ of $[A,X]$ (when A is a co-H-space) is introduced and some of its basic properties developed. As is well known, $G(A,X)$ is not a functor of X but is a contravariant functor of A from the subcategory of H-cogroups and co-H-maps into the category of groups and homomorphisms. We show in Section 2.4 that both $G(X,X)$ and $C(X,X)$ are rings if X is an H-cogroup. In the course of doing so, we also prove that $G(A,X)$ is a contravariant functor of A from the full subcategory of H-cogroups and maps (not necessarily co-H-maps) into the category of abelian groups and homomorphisms.

2.2 Certain Basic Properties of $G(A,X)$

The purpose of this section is to record two basic results, an application of the first was already demonstrated in Section 1.4 of the previous chapter. First we recall the following well known lemma:

Lemma 2.2.1 ([21]). Let X be a locally compact Hausdorff space, Z a Hausdorff space and Y any space. Then the function spaces $(Y^X)^Z$ and $Y^{X\times Z}$ are homeomorphic and a homeomorphism $H: (Y^X)^Z \to Y^{X\times Z}$ is given by $H(g)(x,z) = g(z)(x)$ for each $g: Z \to Y^X$, $x \in X$, $z \in Z$. Furthermore, $f \simeq g$ iff $H(f) \simeq H(g)$.

Theorem 2.2.2. Let X be a space having the homotopy type of a locally finite CW-complex and A any Hausdorff space. Suppose $\omega: X^X \to X$ is the evaluation map where X^X is the space of free maps from X to X with 1_X as base point. Then $\omega_{\#}([A,X^X]) = G(A,X)$ as sets, where $\omega_{\#}$ is the induced function of ω.

Proof. Let $[g] \in [A,X^X]$. Let H be the homeomorphism given in Lemma 2.2.1. We claim that $H(g)j = \nabla(1\vee\omega g)$ where $j: X \vee A \to X \times A$ is the inclusion. Indeed, for each $x \in X$ and $a \in A$, we have $H(g)j(x,*) = H(g)(x,*) = g(*)(x) = 1_X(x) = x$, $\nabla(1\vee\omega g)(x,*) = \nabla(x,\omega g(*)) = x$, $H(g)j(*,a) = H(g)(*,a) = g(a)(*) = \omega(g(a))$ and $\nabla(1\vee\omega g)(*,a) = \nabla(*,\omega(g(a))) = \omega(g(a))$. Thus $H(g)j = \nabla(1\vee\omega g)$, so that $\omega_{\#}[g] = [\omega g] \in G(A,X)$. Hence $\omega_{\#}([A,X^X]) \subset G(A,X)$.

Conversely, let $f \in G(A,X)$. Then there exists a map $F: X \times A \to X$ such that $Fj = \nabla(1\vee f)$. By Lemma 2.2.1, we can find a map $f': A \to X^X$ such that $F = H(f')$. Then $\omega f'(a) = \omega(f'(a)) = f'(a)(*) = H(f')(*,a) = F(*,a) = f(a)$. Thus $\omega_{\#}[f'] = [\omega f'] = [f]$, so that $G(A,X) \subset \omega_{\#}([A,X^X])$. Hence $\omega_{\#}([A,X^X]) = G(A,X)$.

Under the same hypotheses as the above theorem, if, in addition, A is an H-cogroup, then $\omega_{\#}([A,X^X]) = G(A,X)$ as groups.

This justifies the term evaluation subgroup.

Remark 1. If $A = S^1$, then we have $G(X) = \omega_\#(\pi_1(X^X))$ which is Theorem III.1 of [9].

Remark 2. If $A = S^n$, then we have $G_n(X) = \omega_\#(\pi_n(X^X))$ which is Proposition 1.1 of [12].

Next we shall prove a product theorem which yields several corollaries, including another result of [12].

Theorem 2.2.3. Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a collection of spaces which have the homotopy type of CW-complexes and A any space. Then $G(A,\pi X_\alpha)$ and $\pi G(A,X_\alpha)$ are isomorphic as sets, where π denotes the topological product or set product as the case may be.

Proof. Let $f \in G(A,\pi X_\alpha)$. Then there exists a map $F\colon \pi X_\alpha \times A \to \pi X_\alpha$ such that $Fj = \nabla(1\vee f)$ where j is the obvious inclusion. For each $\beta \in \Lambda$, let $f_\beta\colon A \to X_\beta$ be the map given by $f_\beta = p_\beta f$ where $p_\beta\colon\pi X_\alpha \to X_\beta$ is the obvious projection. We claim that $f_\beta \in G(A,X_\beta)$. To see this, let $F_\beta = p_\beta F(i_\beta\times 1)\colon X_\beta \times A \to X_\beta$ where $i_\beta\colon X_\beta \to \pi X_\alpha$ is the inclusion. If j_β denotes the inclusion $X_\beta \vee A \to X_\beta \times A$, then $F_\beta j_\beta = p_\beta F(i_\beta\times 1)j_\beta = p_\beta Fj(i_\beta\vee 1) = p_\beta\nabla(1\vee f)(i_\beta\vee 1)$ $= \nabla(p_\beta\vee p_\beta)(i_\beta\vee f) = \nabla(p_\beta i_\beta\vee p_\beta f) = \nabla(1\vee f_\beta)$. Hence $f_\beta \in G(A,X_\beta)$. We may therefore define a function $\Phi\colon G(A,\pi X_\alpha) \to \pi G(A,X_\alpha)$ as follows: for each $f \in G(A,\pi X_\alpha)$, let $\Phi(f) = \langle f_\alpha\rangle$ where $f_\alpha = p_\alpha f$ for each α.

Conversely, for each $\beta \in \Lambda$, let $f_\beta \in G(A,X_\beta)$. Then we can find a map $F_\beta\colon X_\beta \times A \to X_\beta$ such that $F_\beta j_\beta = \nabla(1\vee f_\beta)$ where j_β

is the obvious inclusion. Define a map $f\colon A \to \pi X_\alpha$ by $f = (<f_\alpha>)\Delta$. We claim that $f \in G(A,\pi X_\alpha)$. In fact, let $F\colon \pi X_\alpha \times A \to \pi X_\alpha$ be the map given by $F(<x_\alpha>,a) = <F_\alpha(x_\alpha,a)>$ for each $<x_\alpha> \in \pi X_\alpha$ and $a \in A$. Then $Fj(<x_\alpha>,*) = <F_\alpha(x_\alpha,*)> = <x_\alpha> = \nabla(1\vee f)(<x_\alpha>,*)$ and $Fj(*,a) = <F_\alpha(*,a)> = <f_\alpha(a)> = f(a) = \nabla(1\vee f)(*,a)$. Thus $Fj = \nabla(1\vee f)$ and hence $f \in G(A,\pi X_\alpha)$. We may therefore define a function $\Psi\colon \pi G(A,X_\alpha) \to G(A,\pi X_\alpha)$ as follows: for each $<f_\alpha> \in \pi G(A,X_\alpha)$, let $\Psi(<f_\alpha>) = (<f_\alpha>)\Delta$. Moreover, it can be easily verified that the functions Φ and Ψ are inverse to each other and this establishes a one-to-one correspondence between the sets $G(A,\pi X_\alpha)$ and $\pi G(A,X_\alpha)$. The proof of the theorem is thus complete.

An immediate consequence of the above theorem is the following result.

Corollary 2.2.4. Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a collection of spaces which have the homotopy type of CW-complexes and A an H-cogroup. Then $G(A,\pi X_\alpha) \cong \oplus\, G(A,X_\alpha)$ as groups, where $\oplus$ denotes the direct product.

Proof. In view of the preceding theorem, it suffices to show that the function Φ defined in the proof is a homomorphism of groups. To do this, let $f,g \in G(A,\pi X_\alpha)$ and ϕ the given co-H-structure on A. Using the same symbol $+$ for the different group operations, we have

$$\begin{aligned}\Phi(f+g) &= \Phi(\nabla(f\vee g)\phi) \\ &= \langle p_\alpha \nabla(f\vee g)\phi\rangle \\ &= \langle \nabla(p_\alpha \vee p_\alpha)(f\vee g)\phi\rangle \\ &= \langle \nabla(p_\alpha f \vee p_\alpha g)\phi\rangle \\ &= \langle p_\alpha f + p_\alpha g\rangle \\ &= \Phi(f) + \Phi(g)\ .\end{aligned}$$

Hence Φ provides the indicated isomorphism.

In particular, we have the following corollary which includes Theorem 2.1 of [12] as a special case.

Corollary 2.2.5. Let X and Y be spaces which have the homotopy type of CW-complexes and A an H-cogroup. Then $G(A,X\times Y) \cong G(A,X) \oplus G(A,Y)$ as groups, where $\oplus$ denotes the direct product.

We end this section with an observation on $G(A,X)$. Let $\tau\colon [\Sigma A,X] \to [A,\Omega X]$ be the natural isomorphism. Then one might conjecture that τ induces an isomorphism between $G(\Sigma A,X)$ and $G(A,\Omega X)$. But this is not the case. To see this, let $A = S^1$ and $X = S^2$. Then we have $G(\Sigma A,X) = G(S^2,S^2) = o$ and $G(A,\Omega X) = G(S^1,\Omega S^2)$ $= [S^1,\Omega S^2] = Z$.

2.3 Some Basic Properties of $C(A,X)$

In this section we shall define a larger subset $C(A,X)$ of $[A,X]$ which includes $G(A,X)$ and shall study some of its basic properties. As it will be seen from the next section that the introduction of $C(A,X)$ will provide a new insight into $G(A,X)$, it

deserves our attention here.

Definition 2.3.1. Let A be a co-H-space (thus the function $\Omega: [A,X] \to [\Omega A,\Omega X]$, given by $f \mapsto \Omega f$, is injective). We define $C(A,X) \equiv \Omega^{-1}[\Omega A,\Omega X]_{C\Omega}$ where $[\Omega A,\Omega X]_{C\Omega}$ denotes the set of all homotopy classes of maps Ωf which are central.

Theorem 2.3.2. If A is a co-H-space, then $G(A,X) \subset C(A,X)$, and $G(A,\Sigma X) = C(A,\Sigma X)$ for all spaces X.

Proof. We need only show that $C(A,\Sigma X) \subset G(A,\Sigma X)$ since by Lemma 1.3.10, we have $G(A,X) \subset C(A,X)$ for all spaces X. We first consider the case where A is a suspension, say $A = \Sigma B$. Thus suppose $f: \Sigma B \to \Sigma X$ is in $C(\Sigma B,\Sigma X)$, that is, $\Omega f: \Omega\Sigma B \to \Omega\Sigma X$ is central. By Lemma 1.3.9, we have $\nabla(1 \vee f)L \simeq *$ where $L: \Sigma X \flat \Sigma B \to \Sigma X \vee \Sigma B$ is the fibre of the inclusion $\Sigma X \vee \Sigma B \to \Sigma X \times \Sigma B$. Let $i_1: \Sigma X \to \Sigma X \vee \Sigma B$ and $i_2: \Sigma B \to \Sigma X \vee \Sigma B$ be the obvious inclusions. Then we can form the generalized Whitehead product $[i_1,i_2]: \Sigma(X \wedge B) \to \Sigma X \vee \Sigma B$. Since $[i_1,i_2]$ coclassifies $\Sigma X \times \Sigma B$, it follows that $[i_1,i_2]$ factors through L, that is, we can find a map $g: \Sigma(X \wedge B) \to \Sigma X \flat \Sigma B$ such that $Lg \simeq [i_1,i_2]$. Thus $\nabla(1 \vee f)[i_1,i_2] \simeq \nabla(1 \vee f)Lg \simeq *$, and hence we can find a map $h: \Sigma X \times \Sigma B \to X$ which extends $\nabla(1 \vee f)$. Thus $f \in G(\Sigma B,\Sigma X)$. In the general case where A is a co-H-space, let $e: \Sigma\Omega A \to A$ be the adjoint of the identity map $\Omega A \to \Omega A$. Consider the map $fe: \Sigma\Omega A \to \Sigma X$. Since Ωf is central, so is $\Omega(fe) = (\Omega f)(\Omega e)$ by Lemma 1.3.8. Since the domain of fe is a suspension, by the first part of the proof, it follows that fe is cyclic. Since A is a

co-H-space, we can find a map $s: A \to \Sigma\Omega A$ such that $es \simeq 1$. Hence $fes \simeq f$ is cyclic.

Remark. In the case that A is a suspension, this theorem has been proved by Hoo in the following form.

Corollary 2.3.3 ([20]). Let $f: \Sigma B \to \Sigma X$ be a map. Then the following are equivalent:

(a) f is cyclic.

(b) f maps $\Omega\Sigma B$ into the center of $\Omega\Sigma X$.

(c) $[1_{\Sigma X}, f] = o$, where $[,]$ denotes the generalized Whitehead product.

Remark. Note that condition (b) simply means that Ωf is central.

We shall show in the next section that $C(A,X)$ is a subgroup contained in the center of $[A,X]$ if A is a co-H-space with a right homotopy inverse and X is any space.

Definition 2.3.4.

$$W(\Sigma A,X) \equiv \{\alpha \in [\Sigma A,X] \mid [\alpha,\beta] = o \text{ for all } \beta \in [\Sigma B,X] \text{ and for all } B\}$$

$$P(\Sigma A,X) \equiv \{\alpha \in [\Sigma A,X] \mid [\alpha,\beta] = o \text{ for all } \beta \in [\Sigma^{\ell} A,X] \text{ and for all } \ell \geq 1\}.$$

Here $[\alpha,\beta]$, as usual, denotes the generalized Whitehead product of α and β.

Clearly $W(\Sigma A,X) \subset P(\Sigma A,X)$. It is shown in [31] that $P(\Sigma A,X)$ is a subgroup of $[\Sigma A,X]$. We now relate $C(\Sigma A,X)$ and

$W(\Sigma A,X)$ for any spaces A and X.

Proposition 2.3.5. Let A, B and X be spaces. If $f \in C(\Sigma A,X)$ then $[f,g] = o$ for all $g \in [\Sigma B,X]$.

Proof. Let $q\colon A \times B \to A \wedge B$ be the quotient map, and let $p_1\colon A \times B \to A$, $p_2\colon A \times B \to B$ be the usual projections. Then according to [1], we have

$$[f,g]\Sigma q = f\Sigma p_1 + g\Sigma p_2 - f\Sigma p_1 - g\Sigma p_2 \ .$$

Taking adjoints, we obtain the equation

$$\tau([f,g])q = (\Omega f)e_1'p_1 + (\Omega g)e_2'p_2 - (\Omega f)e_1'p_1 - (\Omega g)e_2'p_2$$

where $e_1'\colon A \to \Omega\Sigma A$, $e_2'\colon B \to \Omega\Sigma B$ are the adjoints of the obvious identity maps. Since Ωf is central, it follows that $\tau([f,g])q = o$. Now $q^{\#}$ is a monomorphism, and τ, the operation of taking adjoints, is an isomorphism. Hence $[f,g] = o$.

Corollary 2.3.6. For all spaces A and X, we have

$$G(\Sigma A,X) \subset C(\Sigma A,X) \subset W(\Sigma A,X) \subset P(\Sigma A,X) \subset [\Sigma A,X] \ .$$

Theorem 2.3.7. For all spaces A and X, we have

$$G(\Sigma A,\Sigma X) = C(\Sigma A,\Sigma X) = W(\Sigma A,\Sigma X) \ .$$

Proof. We need only show that $W(\Sigma A,\Sigma X) \subset G(\Sigma A,\Sigma X)$. Let $f \in W(\Sigma A,\Sigma X)$. Then $[f,1_{\Sigma X}] = o$ by Definition 2.3.4, so that $[1_{\Sigma X},f] = o$. According to Corollary 2.3.3, $f \in G(\Sigma A,\Sigma X)$.

<u>Theorem 2.3.8.</u> For any space X, we have

$$G(\Sigma X,\Sigma X) = C(\Sigma X,\Sigma X) = W(\Sigma X,\Sigma X) = P(\Sigma X,\Sigma X),$$

(see the example following Theorem 2.4.7).

<u>Proof.</u> It is obvious that $W(\Sigma X,\Sigma X) = P(\Sigma X,\Sigma X)$.

As an application of Proposition 2.3.5, we have the following result.

<u>Proposition 2.3.9.</u> Let G be a topological group and H a closed subgroup. Let $p: G \to G/H$ be the natural map onto the space of left cosets. If A is such that $p_{\#}: [\Sigma A,G] \to [\Sigma A,G/H]$ is onto, then for all α in $[\Sigma A,G/H]$ and all β in $[\Sigma B,G/H]$ where B is any space, we have $[\alpha,\beta] = o$.

<u>Proof.</u> Since $p_{\#}$ is onto, we can find a map $\gamma: \Sigma A \to G$ such that $p\gamma = \alpha$. Then α is cyclic, and the assertion follows from Proposition 2.3.5.

<u>Remark.</u> The above proposition says that

$$W(\Sigma A,G/H) = [\Sigma A,G/H] .$$

2.4 <u>$G(X,X)$ and $C(X,X)$ as Rings</u>

We have now come to the central part of the chapter. Our main object here is to show that both $G(X,X)$ and $C(X,X)$ are rings if X is an H-cogroup. In the course of achieving our aim, we show that for a fixed space X both $G(-,X)$ and $C(-,X)$ turn out to be contravariant functors from the <u>full subcategory</u> of

<u>H-cogroups</u> and <u>maps</u> (not necessarily co-H-maps) into the category of <u>abelian groups</u> and <u>homomorphisms</u>.

We shall first prove the following theorem.

<u>Theorem 2.4.1</u>. Let A be a co-H-space with a right homotopy inverse ν, and let X be a space. Then $C(A,X)$ and $[\Omega A,\Omega X]_{C\Omega}$ are subgroups contained in the centers of $[A,X]$ and $[\Omega A,\Omega X]$ respectively, and $\Omega\colon C(A,X) \to [\Omega A,\Omega X]_{C\Omega}$ is an isomorphism of abelian groups.

To show this, we have to appeal to a result of Hoo [19]. Consider a co-H-space A with co-H-structure $\phi\colon A \to A \vee A$. Applying the co-Hopf construction to ϕ, we obtain a map $H(\phi)\colon \Omega A \to \Omega(A \flat A)$. Let $f,g\colon A \to X$ be maps. Let $L\colon A \flat A \to A \vee A$ be the fibre of $A \vee A \to A \times A$. Then we can form $\Omega\{\nabla(f\vee g)L\}H(\phi)\colon \Omega A \to \Omega X$.

<u>Lemma 2.4.2 ([19])</u>. Let A be a co-H-space with co-H-structure $\phi\colon A \to A \vee A$. Let $f,g\colon A \to X$ be maps. Then $\Omega(f+g) = \Omega\{\nabla(f\vee g)L\}H(\phi) + \Omega f + \Omega g$.

<u>Proof of Theorem</u>. According to Lemma 1.3.9, if f or h is in $C(A,X)$ then $\nabla(f\vee h)L \simeq *$ and hence $\Omega(f+h) = \Omega f + \Omega h$. Let $\Omega f,\Omega g \in [\Omega A,\Omega X]_{C\Omega}$. Then $\Omega(f+g) = \Omega f + \Omega g$. By parts (b) and (c) of Lemma 1.3.8, both $(\Omega f)p_2$ and $(\Omega g)p_2$ lie in the center of $[\Omega X\times\Omega A,\Omega X]$, so that $(p_1,(\Omega f+\Omega g)p_2) = (p_1,(\Omega f)p_2 + (\Omega g)p_2) = o$. According to part (a) of Lemma 1.3.8, $\Omega f + \Omega g$ is central. Let $\mu\colon \Omega X \to \Omega X$ be the loop inverse. We shall show that $-\Omega f = \mu\Omega f$ is central. In fact, since $o = \Omega(f+f\nu) = \Omega f + \Omega(f\nu)$, $-\Omega f = (\Omega f)(\Omega\nu)$ is central. Thus $[\Omega A,\Omega X]_{C\Omega}$ is a subgroup of

$[\Omega A, \Omega X]$. That it is contained in the center is clear. To see that $C(A,X)$ is contained in the center of $[A,X]$, let $h \in [A,X]$. Then $\Omega f + \Omega h = \Omega h + \Omega f$ since $\Omega f \in [\Omega A, \Omega X]_{C\Omega}$. Thus $\Omega(f+h) = \Omega(h+f)$ and hence $f+h = h+f$ since $\Omega\colon [A,X] \to [\Omega A, \Omega X]$ is injective. Hence $C(A,X)$ and $[\Omega A, \Omega X]_{C\Omega}$ are subgroups contained in the centers of $[A,X]$ and $[\Omega A, \Omega X]$ respectively, and $\Omega\colon C(A,X) \to [\Omega A, \Omega X]_{C\Omega}$ is an isomorphism of abelian groups. This completes the proof of the theorem.

Remark. It follows that if A is a co-H-space with a right homotopy inverse, then for every space X, $G(A,X) \subset C(A,X) \subset$ center of $[A,X]$ as subgroups. This generalizes Gottlieb's result [9] that $G(X)$ lies in the center of $\pi_1(X)$.

We shall now proceed to establish the right distributive law. Suppose that $f\colon A \to B$ is a map from a homotopy associative co-H-space A to a co-H-space B. Then we can find a co-H-map $s\colon A \to \Sigma\Omega A$ such that $es \simeq 1$ where $e\colon \Sigma\Omega A \to A$ is the usual map. Let $g_1, g_2\colon B \to Y$ be maps where Y is any space. We can form $(g_1+g_2)f\colon A \to Y$. In general $(g_1+g_2)f \neq g_1 f + g_2 f$. A suitable distributive law would compensate for this by providing a correction term. For our purposes, the correction term would have to be such that it vanishes in case g_1 or g_2 is in $C(B,Y)$.

Consider the map $\phi f\colon A \to B \vee B$ where ϕ is the co-H-structure on B. Applying the co-Hopf construction to this map, we obtain a map $H(\phi f)\colon \Omega A \to \Omega(B \flat B)$. Taking adjoint, we obtain $\tau^{-1}\{H(\phi f)\}\colon \Sigma\Omega A \to B \flat B$. Let $L\colon B \flat B \to B \vee B$ be the

fibre of $B \vee B \to B \times B$. Then we can form $L\tau^{-1}\{H(\phi f)\}s\colon A \to B \vee B$. The suitable distributive law may now be stated.

Lemma 2.4.3 ([19]). Let $f\colon A \to B$ be a map from a homotopy associative co-H-space A to a co-H-space B, and let $s\colon A \to \Sigma\Omega A$ be a co-H-map such that $es \simeq 1\colon A \to A$. Let $\phi\colon B \to B \vee B$ be the co-H-structure on B. Let $g_1, g_2\colon B \to Y$ be maps, where Y is any space. Then

$$(g_1+g_2)f = \nabla(g_1 \vee g_2)L\tau^{-1}\{H(\phi f)\}s + g_1 f + g_2 f \ .$$

Theorem 2.4.4. Let $f\colon A \to B$ be a map from a homotopy associative co-H-space A to a co-H-space B. Let $g_1, g_2\colon B \to Y$ be maps such that either g_1 or g_2 is in $C(B,Y)$, where Y is any space. Then $(g_1+g_2)f = g_1 f + g_2 f$.

Proof. According to Lemma 1.3.9, $\nabla(g_1 \vee g_2)L \simeq *$ and hence the relation in Lemma 2.4.3 reduces to $(g_1+g_2)f = g_1 f + g_2 f$ as asserted.

Remark. If f is a co-H-map, then the above theorem is trivial.

Theorem 2.4.4 yields the following corollaries.

Corollary 2.4.5. Let $f\colon A \to B$ be a map from a homotopy associative co-H-space A to a co-H-space B. Then $f^{\#}\colon C(B,X) \to C(A,X)$ and $f^{\#}\colon G(B,X) \to G(A,X)$ are homomorphisms for any space X.

Proof. The first part follows directly from Theorem 2.4.4. Restriction to $G(B,X)$ gives the other result.

Corollary 2.4.6. If $f\colon \Sigma A \to B$ is a map where B is a co-H-space, then $f^{\#}\colon C(B,X) \to C(\Sigma A,X)$ and $f^{\#}\colon G(B,X) \to G(\Sigma A,X)$ are homomorphisms for any space X.

Example 1. Let $f\colon \Sigma A \to \Sigma B$ be any map. Then $f^{\#}\colon C(\Sigma B,X) \to C(\Sigma A,X)$ and $f^{\#}\colon G(\Sigma B,X) \to G(\Sigma A,X)$ are group homomorphisms for any space X.

Example 2. Let $f\colon S^n \to S^r$ be any map. Then $f^{\#}\colon C(S^r,X) \to C(S^n,X)$ and $f^{\#}\colon G_r(X) \to G_n(X)$ are group homomorphisms for any space X.

In view of Theorems 2.4.1 and 2.4.4, we conclude that both $G(-,X)$ and $C(-,X)$ are contravariant functors from the full subcategory of H-cogroups and maps into the category of abelian groups and homomorphisms.

Remark. Without Theorems 2.4.1 and 2.4.4 the above observation would be by no means trivial although it is evident from the remark following Theorem 2.4.4 that $G(-,X)$ is a contravariant functor from the subcategory of H-cogroups and co-H-maps into the category of groups and homomorphisms.

The following theorem is now clear.

Theorem 2.4.7. For any H-cogroup X, $G(X,X)$ and $C(X,X)$ are rings.

Example. For any space X,

$$G(\Sigma X,\Sigma X) = C(\Sigma X,\Sigma X) = W(\Sigma X,\Sigma X) = P(\Sigma X,\Sigma X)$$

as rings (see Theorem 2.3.8).

CHAPTER III

COCYCLIC MAPS

3.1 Introduction

The results contained in the present chapter can be regarded as dual to those of Chapter I. As in Section 1.2 of Chapter I, Section 3.2 of this chapter deals with the definition and the existence of cocyclic maps. In Section 3.3, we show that cocyclicity of maps is closed under the wedge product. The dual notion of centrality of maps is also considered and some of its basic properties derived.

3.2 Definition and Existence of Cocyclic Maps

In this section, we dualize the notion of cyclicity of maps. This dual notion was first defined and studied in considerable detail in [31]. We now recall the definition.

Definition 3.2.1 ([31]). A map $f: X \to A$ is said to be *cocyclic* if we can find a map $\phi: X \to X \vee A$ such that the following diagram is homotopy commutative: that is $j\phi \simeq (1\times f)\Delta$. We call such a map ϕ a *coassociated map* of f. The set of all homotopy classes of cocyclic maps from X to A is denoted by $DG(X,A)$.

$$\begin{tikzcd} & X \times A \\ X \arrow[ur, "(1\times f)\Delta"] \arrow[r, "\phi"'] \arrow[ur, phantom, shift right=4ex, "\simeq"] & X \vee A \arrow[u, "j"'] \end{tikzcd}$$

Remark 1. In general, $DG(X,A)$ is only a set, but it turns out to be an abelian group if A is an H-group (see the next chapter).

Remark 2. If $A = K(\pi,n)$, an Eilenberg-MacLane complex of type (π,n), then $DG(X,A)$ reduces to $G^n(X;\pi)$ ([15]) which is called the n^{th} coevaluation subgroup of X (with respect to π).

Lemma 3.2.2 ([31]). If $f: X \to A$ is a cocyclic map and $\theta: A \to B$ is an arbitrary map, then the map $\theta f: X \to B$ is cocyclic.

The existence of cocyclic maps is easily seen from the next proposition.

Proposition 3.2.3. Let X be a space. Then the following are equivalent:

(a) X is a co-H-space.

(b) 1_X is cocyclic.

(c) $DG(X,A) = [X,A]$ for any space A.

Proof. The proof is exactly dual to that of Proposition 1.2.3.

Corollary 3.2.4 ([31]). If X is a co-H-space, then any map $f: X \to A$ is cocyclic.

Dual to cyclic maps, we can obtain cocyclic maps from cofibrations. Suppose that $X \xrightarrow{f} Y \xrightarrow{q} C$ is a cofibration. Then it gives rise to a Puppe sequence:

$$X \xrightarrow{f} Y \xrightarrow{q} C \xrightarrow{\partial} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \longrightarrow \cdots .$$

We have a cooperation $\phi: C \to C \vee \Sigma X$ such that $j\phi \simeq (1\times\partial)\Delta$. Thus ∂ is cocyclic and hence $\partial^{\#}([\Sigma X,A]) \subset DG(C,A)$ for all spaces A.

As an application of this fact, we have the following result.

Theorem 3.2.5. Let $\Omega A \xrightarrow{f} Y \longrightarrow C$ be a cofibration. If $DG(C,A) = o$, then there exists a map $\rho: Y \to \Omega A$ such that $\rho f \simeq 1_{\Omega A}$.

Proof. Since $DG(C,A) = o$, we have $\partial^{\#}([\Sigma\Omega A,A]) = o$, so that $(\Sigma f)^{\#}$ is onto. Thus for $e: \Sigma\Omega A \to A$, we can find a map $g: \Sigma Y \to A$ such that $(\Sigma f)^{\#}[g] = [e]$, that is $g(\Sigma f) \simeq e$. Taking adjoints, we get $\tau(g)f \simeq 1_{\Omega A}$.

3.3 Some Basic Properties of Cocyclic Maps

Let $f: X \to A$ be a cocyclic map. Then if the map $h: Y \to X$ has a left homotopy inverse, $fh: Y \to A$ is also cocyclic. We have also seen that $\theta f: X \to B$ is cocyclic for any map $\theta: A \to B$.

Example. Let A be an H-space and $f: X \to A$ a map. Then f is cocyclic iff $e'f: X \to \Omega\Sigma A$ is cocyclic where $e': A \to \Omega\Sigma A$ is the usual map.

The following lemma shows that cocyclicity of maps is closed under the wedge product.

Lemma 3.3.1. If the maps $f: X \to A$ and $g: Y \to B$ are cocyclic, then so is $f \vee g: X \vee Y \to A \vee B$.

Proof. Let ϕ and ψ be two coassociated maps of f and g respectively. Let

$$\lambda \equiv (1 \vee T \vee 1)(\phi \vee \psi): X \vee Y \to (X \vee Y) \vee (A \vee B) .$$

Then λ is a coassociated map of $f \vee g$.

<u>Definition 3.3.2</u>. Let (G,ϕ,ν) be a H-cogroup and A any space. We say that a map $f: G \to A$ is <u>cocentral</u> if $(1 \vee f)c \simeq *$ where $c : G \to G \vee G$ is the basic cocommutator map (that is, $c \equiv \nabla(1 \vee 1 \vee \nu \vee \nu)(\phi \vee \phi)\phi$).

The following lemmas are immediate consequences of the definition.

<u>Lemma 3.3.3</u>. If $f: G \to A$ is a cocentral map and $\theta: A \to B$ is an arbitrary map, then the map $\theta f: G \to B$ is cocentral.

<u>Proof</u>. Since f is cocentral, we have $(1 \vee f)c \simeq *$. Thus $(1 \vee \theta f)c = (1 \vee \theta)(1 \vee f)c \simeq *$ and hence θf is cocentral.

<u>Lemma 3.3.4</u>. Let $i_1: G \to G \vee A$ and $i_2: A \to G \vee A$ be inclusions. Then $f \in [G,A]$ is cocentral iff $(i_1,i_2 f) = o \in [G,G \vee A]$.

<u>Proof</u>. Let $j_1,j_2: G \to G \vee G$ be the obvious inclusions. Then

$$\begin{aligned}(1 \vee f)c &= (1 \vee f)\{\phi+(\nu \vee \nu)\phi\} \\ &= (1 \vee f)\phi + (\nu \vee f\nu)\phi \\ &= (1 \vee f)(j_1+j_2) + (\nu \vee f\nu)(j_1+j_2) \\ &= (i_1+i_2 f) + (-i_1-i_2 f) \\ &= (i_1,i_2 f) \quad .\end{aligned}$$

Hence the assertion follows.

<u>Lemma 3.3.5</u>. Any cocentral map $f: G \to A$ lies in the center of $[G,A]$.

<u>Proof</u>. We first note that $(1\vee f)c \simeq *$ iff $(1\vee f)\phi = T(f\vee 1)\phi$ where T is the switching map. Let $g \in [G,A]$. Then

$$\begin{aligned}
\nabla(f\vee g)\phi &= \nabla(1\vee g)(f\vee 1)\phi \\
&= \nabla(g\vee 1)T(f\vee 1)\phi \\
&= \nabla(g\vee 1)(1\vee f)\phi \\
&= \nabla(g\vee f)\phi.
\end{aligned}$$

Thus $f+g = g+f$ for all $g \in [G,A]$ and the assertion follows.

<u>Lemma 3.3.6 ([7])</u>. Let $f\colon X \to A$ be a map. Then Σf is cocentral iff $e'q(1\times f)\Delta \simeq *$ where $e'\colon X \wedge A \to \Omega\Sigma(X\wedge A)$ is the usual map and $q\colon X \times A \to X \wedge A$ is the quotient map.

<u>Corollary 3.3.7</u>. If $f\colon X \to A$ is a cocyclic map, then the map Σf is cocentral.

<u>Proof</u>. This follows from the above lemma and the existence of a map $\phi\colon X \to X \vee A$ such that $j\phi \simeq (1\times f)\Delta$. In fact, we have $e'q(1\times f)\Delta \simeq e'qj\phi \simeq *$ from the following homotopy commutative diagram.

$$\begin{tikzcd}
 & X \times A \arrow[r, "q"] & X \wedge A \arrow[r, "e'"] & \Omega\Sigma(X\wedge A) \\
X \arrow[ur, "(1\times f)\Delta"] \arrow[r, "\phi"'] \arrow[ur, phantom, "\simeq"', shift right=2ex] & X \vee A \arrow[u, "j"'] & &
\end{tikzcd}$$

We conclude this chapter with a remark. While we were able to investigate the relationship between cyclicity of maps and maps of finite order in Section 1.4 of Chapter I, we do not know

whether those results can be dualized. The main obstacle for obtaining the dual results is our lack of knowledge about the dual of $G(A,X) = \omega_{\#}([A,X^X])$ and those of the results of Gottlieb on homology ([14]).

CHAPTER IV

COEVALUATION SUBGROUPS

4.1 Introduction

Basically the results contained in this chapter are dual to those of Chapter II. However, the proofs of some of these results, for instance that of Theorem 4.2.1, are quite different from those of Chpater II. Furthermore, there still remain many open questions (see the remarks after Definitions 4.2.6 and 4.2.9) in the dual case. In Section 4.2, we show that $DG(X,A)$ is a subgroup (in fact, abelian) of $[X,A]$ when A is an H-group, thus settling a problem of Varadarajan [31]. We also dualize a result of Chapter II. Using the notion and properties of cocentrality introduced and developed in Chapter III, we show in Section 4.3 that $DC(X,A)$ and $DG(X,A)$ are covariant functors of A from the full subcategory of H-groups and maps into the category of abelian groups and homomorphisms. From this we deduce that both $DC(X,X)$ and $DG(X,X)$ are rings for any H-group X.

4.2 Some Basic Properties of $DG(X,A)$ and $DC(X,A)$

In this section, we shall derive some basic properties of $DG(X,A)$ and $DC(X,A)$ which are the duals of $G(A,X)$ and $C(A,X)$ respectively. Some other properties will also be discussed in the next section. Among other things, we settle a problem of Varadarajan [31] by showing that $DG(X,A)$ is a subgroup of $[X,A]$ when A is an H-group. Thus we can naturally call $DG(X,A)$ the

coevaluation subgroup of X with respect to A when it is a subgroup.

<u>Theorem 4.2.1</u>. If A is an H-group, then $DG(X,A)$ is a subgroup (in fact, it is abelian, see Theorem 4.3.2) of $[X,A]$ for any space X.

<u>Proof</u>. Let m and μ be the H-structure and the inverse on A respectively. Then the inverse of f in the group $[X,A]$ is the homotopy class of $\mu f\colon X \to A$. According to Lemma 3.2.2, $\mu f \in DG(X,A)$ if $f \in DG(X,A)$. Hence $DG(X,A)$ is closed under inversion. To see that it is closed under the operation $+$ in $[X,A]$, let $f,g \in DG(X,A)$. Then we can find maps $\phi,\psi\colon X \to X \vee A$ such that $j\phi \simeq (1\times f)\Delta$ and $j\psi \simeq (1\times g)\Delta$. Let $i\colon (X\vee A) \vee A \to X \vee (A\times A)$ and $i'\colon (X\times A) \vee A \to X \times (A\times A)$ be the obvious inclusions. Then we have the following homotopy commutative diagram:

$$\begin{tikzcd}
 & & (X\vee A)\vee A \arrow[r, "i"] \arrow[d, "j\vee 1"] \arrow[dr, phantom, "\copyright"] & X\vee(A\times A) \arrow[r, "1\vee m"] \arrow[d, "j_2"] \arrow[dr, phantom, "\copyright"] & X\vee A \arrow[d, "j"] \\
 & X\vee A \arrow[ur, "\phi\vee 1"] \arrow[r, "(1\times f)\Delta\vee 1"'] \arrow[d, "j"] \arrow[ur, phantom, "\simeq", shift right=6] \arrow[dr, phantom, "\copyright"] & (X\times A)\vee A \arrow[r, "i'"] \arrow[d, "j_1"] \arrow[dr, phantom, "\copyright"] & X\times(A\times A) \arrow[r, "1\times m"] & X\times A \\
X \arrow[ur, "\psi"] \arrow[r, "(1\times g)\Delta"'] \arrow[ur, phantom, "\simeq", shift right=6] & X\times A \arrow[r, "(1\times f)\Delta\times 1"'] & (X\times A)\times A \arrow[urr, "1\times m"'] & &
\end{tikzcd}$$

Here all vertical arrows are inclusions. Let $\lambda \equiv (1\vee m)i(\phi\vee 1)\psi$. Then we have

$$\begin{aligned} j\lambda &\simeq (1\times m)\ \{(1\times f)\Delta\times 1\}\{(1\times g)\Delta\} \\ &= (1\times m)\{(1\times f)\Delta\times g\}\Delta \\ &= \{1\times m(f\times g)\Delta\}\Delta \\ &= \{1\times(f+g)\}\Delta\ . \end{aligned}$$

Thus $f+g \in DG(X,A)$, so that $DG(X,A)$ is closed under $+$. Hence $DG(X,A)$ is a subgroup of $[X,A]$.

Our next result is not quite as trivial as it might first appear.

Proposition 4.2.2. Let X and Y be spaces having the homotopy type of CW-complexes and A any space. Then the sets $DG(X\vee Y,A)$ and $DG(X,A) \times DG(Y,A)$ are isomorphic where $\times$ denotes the cartesian product.

Proof. Let $f \in DG(X\vee Y,A)$. Then we can find a map $\gamma\colon X \vee Y \to (X\vee Y) \vee A$ such that $j\gamma \simeq (1\times f)\Delta$ where $j\colon (X\vee Y) \vee A \to (X\vee Y) \times A$ is the inclusion and $\Delta\colon X \vee Y \to (X\vee Y) \times (X\vee Y)$ is the diagonal map. Let $f_1 = fi_1\colon X \to A$ and $f_2 = fi_2\colon Y \to A$ where $i_1\colon X \to X \vee Y$ and $i_2\colon Y \to X \vee Y$ are inclusions. We claim that $f_1 \in DG(X,A)$ and $f_2 \in DG(Y,A)$. To see this, let $\alpha = p_{13}\gamma i_1$ and $\beta = p_{23}\gamma i_2$ where $p_{13}\colon (X\vee Y) \times A \to X \times A$ is the projection onto the first and the third coordinates and $p_{23}\colon (X\vee Y) \times A \to Y \times A$ is the projection onto the second and the third coordinates. Clearly α and β are maps into $X \vee A$ and $Y \vee A$ respectively. Consider the following homotopy commutative diagram:

$$\begin{tikzcd}
X \arrow[d, "i_1"'] & & \\
X \vee Y \arrow[r, "\gamma"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\simeq"] & (X\vee Y)\vee A \arrow[r, "p_{13}"] \arrow[d, "j"'] \arrow[dr, phantom, "C"] & X \vee A \arrow[d, "j_1"'] \\
(X\vee Y)\times(X\vee Y) \arrow[r, "1\times f"'] & (X\vee Y)\times A \arrow[r, "p_{13}"'] & X \times A
\end{tikzcd}$$

Here j_1 is the obvious inclusion. Note that we also write p_{13} for $p_{13} \mid (X\vee Y) \vee A$. Let $\Delta'\colon X \to X \times X$ be the diagonal map. Then we have $(1\times f_1)\Delta' = (1\times fi_1)\Delta' = p_{13}(i_1\times fi_1)\Delta' = p_{13}(1\times f)\Delta i_1 \simeq j_1 p_{13}\gamma i_1 = j_1\alpha$. Hence $f_1 \in DG(X,A)$. Similarly, we can show that $f_2 \in DG(Y,A)$. We may therefore define a function $\Phi\colon DG(X\vee Y,A) \to DG(X,A) \times DG(Y,A)$ as follows: for each $f \in DG(X\vee Y,A)$, let $\Phi(f) = (f_1,f_2)$ where $f_k = fi_k$, $k = 1,2$.

Conversely, let $f_1 \in DG(X,A)$ and $f_2 \in DG(Y,A)$ be given. Then we can find maps $\alpha\colon X \to X \vee A$ and $\beta\colon Y \to Y \vee A$ such that $j_1\alpha \simeq (1\times f_1)\Delta'$ and $j_2\beta \simeq (1\times f_2)\Delta''$ where j's and Δ's are the obvious inclusions and diagonal maps respectively. Let $f = \nabla(f_1 \vee f_2)\colon X \vee Y \to A$. Then Lemma 3.3.1 could be applied to show that $f \in DG(X\vee Y,A)$. As the proof of Lemma 3.3.1 was only sketched, we would rather provide a direct proof of $f \in DG(X\vee Y,A)$ here. To do this, consider the following homotopy commutative diagram:

$$
\begin{tikzcd}
X \vee Y \arrow[r, "\alpha \vee \beta"] \arrow[d, "\Delta' \vee \Delta''"] \arrow[dd, bend right, "\Delta"'] \arrow[dr, phantom, "\sim"] & (X \vee A) \vee (Y \vee A) \arrow[r, "1 \vee T \vee 1"] \arrow[d, "j_1 \vee j_2"] \arrow[dr, phantom, "c"] & (X \vee Y) \vee (A \vee A) \arrow[r, "1 \vee \nabla"] \arrow[d, "j_3"] \arrow[dr, phantom, "c"] & (X \vee Y) \vee A \arrow[d, "j_4"] \\
(X \times X) \vee (Y \times Y) \arrow[r, "(1 \times f_1) \vee (1 \times f_2)"'] \arrow[d, "1 \times T \times 1"] \arrow[dr, phantom, "c"] & (X \times A) \vee (Y \times A) \arrow[r, "1 \times T \times 1"'] & (X \vee Y) \times (A \vee A) \arrow[r, "1 \times \nabla"'] & (X \vee Y) \times A \\
(X \vee Y) \times (X \vee Y) \arrow[uurr, "1 \times (f_1 \vee f_2)"'] & & &
\end{tikzcd}
$$

Let $\gamma = (1\vee\nabla)(1\vee T\vee 1)(\alpha\vee\beta): X \vee Y \to (X\vee Y) \vee A$. Then

$$j_4\gamma \simeq (1\times\nabla)\ \{1 \times (f_1\vee f_2)\}\Delta$$
$$= \{1 \times \nabla(f_1\vee f_2)\}\Delta$$
$$= (1\times f)\Delta\ .$$

Hence $f \in DG(X\vee Y,A)$. We may therefore define a function

$$\Psi : DG(X,A) \times DG(Y,A) \to DG(X\vee Y,A)$$

as follows: for $f_1 \in DG(X,A)$ and $f_2 \in DG(Y,A)$, let $\Psi(f_1,f_2) = \nabla(f_1\vee f_2)$. Finally, it can be easily verified that Φ and Ψ are inverse to each other and hence they provide the indicated isomorphism. This completes the proof of the proposition.

We are now in a position to establish the following theorem.

Theorem 4.2.3. Let X and Y be spaces having the homotopy type of CW-complexes and A an H-group. Then $DG(X\vee Y,A) \cong DG(X,A) \oplus DG(Y,A)$ as groups, where $\oplus$ denotes the direct product.

Proof. In view of Theorem 4.2.1 and the preceding proposition, it suffices to show that the function Φ defined in the proof of the latter is a homomorphism of groups. To do this, let $f,g \in DG(X\vee Y,A)$ and m the given H-structure on A. Then we have

$$\begin{aligned}
\Phi(f+g) &= \Phi\{m(f\times g)\Delta\} \\
&= (m(f\times g)\Delta i_1, m(f\times g)\Delta i_2) \\
&= (m(fi_1\times gi_1)\Delta, m(fi_2\times gi_2)\Delta) \\
&= (fi_1+gi_1, fi_2+gi_2) \\
&= (fi_1, fi_2) + (gi_1, gi_2) \\
&= \Phi(f) + \Phi(g)\ .
\end{aligned}$$

Hence Φ is a homomorphism of groups.

When $A = K(\pi,n)$, the above theorem reduces to the following result of Haslam.

Corollary 4.2.4 ([15]).

$$G^n(X\vee Y;\pi) = G^n(X;\pi) \oplus G^n(Y;\pi)$$

for all integers $n \geq o$ and abelian groups π.

Example 1. Let T be the torus. Then

$$\begin{aligned}
DG(S^2\vee S^1, T) &\cong DG(S^2, T) \times DG(S^1, T) \\
&= o \times (Z \oplus Z) = Z \oplus Z\ .
\end{aligned}$$

Example 2. Let X be the figure eight space. Then

$$DG(X, S^1) \cong DG(S^1, S^1) \oplus DG(S^1, S^1) = Z \oplus Z\ ,$$

and

$$DG(X, \Omega S^1) = \pi_1(\Omega S^1) \oplus \pi_1(\Omega S^1) = o\ .$$

Let $\{X_\alpha\}$ be a collection of spaces having the homotopy type of CW-complexes. Let $\vee X_\alpha$ be the subspace of the product space πX_α defined as follows:

$$\vee X_\alpha \equiv \{\langle x_\alpha\rangle \mid \text{all coordinates } x_\alpha, \text{ except possibly one, are base points}\}.$$

Then the preceding theorem (resp. proposition) can be extended to the following proposition.

Proposition 4.2.5.

$$DG(\vee X_\alpha, A) \cong \oplus\, DG(X_\alpha, A)$$

as group (resp. as sets), where $\oplus$ denotes the direct product (resp. the cartesian product).

We shall now introduce the subset $DC(X,A)$ of $[X,A]$ which is the dual of $C(A,X)$. If A is an H-space, then the function $\Sigma\colon [X,A] \to [\Sigma X, \Sigma A]$, given by $f \mapsto \Sigma f$, is injective. Let $[\Sigma X, \Sigma A]_{C\Sigma}$ denote the subset of $[\Sigma X, \Sigma A]$ consisting of those homotopy classes of maps Σf which are cocentral.

Definition 4.2.6. Let A be an H-space. We define

$$DC(X,A) \equiv \Sigma^{-1}[\Sigma X, \Sigma A]_{C\Sigma} .$$

Remark. Clearly $DG(X,A) \subset DC(X,A)$ if A is an H-space. The appearance of the factor e' in Lemma 3.3.6 changes some dual results. In particular, it is not known to us whether Theorem 2.3.2 dualizes. However, the distributive laws and Proposition 2.3.5

dualize without any change.

Proposition 4.2.7. Let A, B and X be spaces. If $f \in DC(X,\Omega A)$, then $[f,g]' = o$ for all $g \in [X,\Omega B]$ where $[\ ]'$ is the dual of the generalized Whitehead product.

To show this, we need a lemma.

Lemma 4.2.8. Let $i\colon A \flat B \to A \vee B$ be the inclusion of the flat product into the wedge product of A and B. If X is a co-H-space, then $i_{\#}\colon [X,A\flat B] \to [X,A\vee B]$ is a monomorphism.

Proof of Proposition. Let $i_1\colon A \to A \vee B$ and $i_2\colon B \to A \vee B$ be the usual inclusions. According to [1], we have

$$(\Omega i)[f,g]' = (\Omega i_1)f + (\Omega i_2)g - (\Omega i_1)f - (\Omega i_2)g \quad .$$

Taking τ^{-1}, we obtain

$$i\tau^{-1}([f,g]') = i_1\{e_1(\Sigma f)\} + i_2\{e_2(\Sigma g)\} - i_1\{e_1(\Sigma f)\} \\ - i_2\{e_2(\Sigma g)\} \ ,$$

where $e_1\colon \Sigma\Omega A \to A$ and $e_2\colon \Sigma\Omega B \to B$ are the usual maps. Since Σf is cocentral, so is $i_1\{e_1(\Sigma f)\}$ by Lemma 3.3.3. According to Lemma 3.3.5, we have $i\tau^{-1}([f,g]') = o$. Hence $[f,g]' = o$ as $i_{\#}$ is mono and τ^{-1} is an isomorphism.

Example. Let X be a co-H-space, A and B any spaces. Then $[f,g]' = o$ for all $f \in [X,\Omega A]$ and $g \in [X,\Omega B]$.

Dual to Definition 2.4.3, we have the following definition.

Definition 4.2.9.

$$DW(X,\Omega A) \equiv \{\alpha \in [X,\Omega A] \mid [\alpha,\beta]' = o \text{ for all } \beta \in [X,\Omega B] \text{ and for all } B\}.$$

$$DP(X,\Omega A) \equiv \{\alpha \in [X,\Omega A] \mid [\alpha,\beta]' = o \text{ for all } \beta \in [X,\Omega^{\ell} A] \text{ and for all } \ell \geq 1\}.$$

Remarks. Clearly we have the following inclusions

$$DG(X,\Omega A) \subset DC(X,\Omega A) \subset DW(X,\Omega A) \subset DP(X,\Omega A) \subset [X,\Omega A].$$

It will be interesting to have examples which show that some of the inclusions are proper. It is also not known whether $DP(X,\Omega A)$ is a subgroup of $[X,\Omega A]$ or $DG(\Omega X,\Omega X) = DC(\Omega X,\Omega X) = DW(\Omega X,\Omega X) = DP(\Omega X,\Omega X)$ for any space X.

4.3 DC(X,X) and DG(X,X) as Rings

We first show that $DC(X,A)$ and $DG(X,A)$ are covariant functors of A from the full subcategory of H-groups and maps into the category of abelian groups and homomorphisms. The fact that $DC(X,X)$ and $DG(X,X)$ are rings will then follow immediately. We begin with a lemma.

Lemma 4.3.1 ([19]). Let A be an H-space with H-structure m, and let X be a space. Let $f,g\colon X \to A$ be maps. Then $\Sigma(f+g) = J(m)\Sigma\{q(f\times g)\Delta\} + \Sigma f + \Sigma g$ where $J(m)\colon \Sigma(A\wedge A) \to \Sigma A$ is the Hopf construction on m and $q\colon A \times A \to A \wedge A$ is the quotient map.

Theorem 4.3.2. Let A be an H-space with a right homotopy inverse μ, and let X be a space. Then $DC(X,A)$ and $[\Sigma X,\Sigma A]_{C\Sigma}$ are subgroups contained in the centers of $[X,A]$ and $[\Sigma X,\Sigma A]$ respectively, and $\Sigma: DC(X,A) \to [\Sigma X,\Sigma A]_{C\Sigma}$ is an isomorphism of abelian groups. In particular, $DG(X,A)$ is contained in the center of $[X,A]$.

Proof. We first show that $\Sigma(f+g) = \Sigma f + \Sigma g$ if f or g is in $DC(X,A)$. To do this, consider the map $J(m)\Sigma\{q(f\times g)\Delta\}$. Taking adjoint, we obtain

$$\tau(J(m))\{q(f\times g)\Delta\} = \Omega(J(m))e'q(f\times g)\Delta \simeq * \, .$$

Thus $J(m)\Sigma\{q(f\times g)\Delta\} \simeq *$ and hence $\Sigma(f+g) = \Sigma f + \Sigma g$. With the aid of Lemma 3.3.4, it can be easily verified that $\Sigma f + \Sigma g$ is cocentral for all $f,g \in DC(X,A)$.

Next, we claim that $-\Sigma f$ is cocentral if $f \in DC(X,A)$. In fact,

$$o = \Sigma(f+\mu f) = \Sigma f + \Sigma(\mu f)$$

implies that $-\Sigma f = (\Sigma\mu)(\Sigma f)$ which is cocentral by Lemma 3.3.3. Hence $[\Sigma X,\Sigma A]_{C\Sigma}$ is a subgroup of $[\Sigma X,\Sigma A]$. That $[\Sigma X,\Sigma A]_{C\Sigma}$ lies in the center of $[\Sigma X,\Sigma A]$ follows from Lemma 3.3.5. The facts that $\Sigma(f+g) = \Sigma f + \Sigma g$ if f or g is in $DC(X,A)$ and that Σ is an injection also imply that $DC(X,A)$ lies in the center of $[X,A]$. Consequently, $DC(X,A)$ and $[\Sigma X,\Sigma A]_{C\Sigma}$ are subgroups contained in the centers of $[X,A]$ and $[\Sigma X,\Sigma A]$ respectively, and

$$\Sigma\colon DC(X,A) \to [\Sigma X,\Sigma A]_{C\Sigma}$$

is an isomorphism of abelian groups. The proof of the theorem is thus complete.

Let $f\colon X \to A$ be a map where X is an H-space with H-structure m, and A is a homotopy associative H-space. Then we can find a retraction $\gamma\colon \Omega\Sigma A \to A$ which is an H-map. Suppose $g_1, g_2\colon Y \to X$ are maps where Y is any space. Then we can form $f(g_1+g_2)\colon Y \to A$. Let $J(fm)\colon \Sigma(X\wedge X) \to \Sigma A$ be the Hopf construction on fm. Then we have the following lemma.

Lemma 4.3.3 ([19]).

$$f(g_1+g_2) = \gamma\tau\{J(fm)\}q(g_1\times g_2)\Delta + fg_1 + fg_2 \quad ,$$

where τ is the adjoint functor and $q\colon X \times X \to X \wedge X$ is the quotient map.

Proposition 4.3.4. If $f\colon X \to A$ is a map from an H-space into a homotopy associative H-space A, then

$$f_{\#}\colon DC(Y,X) \to DC(Y,A)$$

and

$$f_{\#}\colon DG(Y,X) \to DG(Y,A)$$

are homomorphisms for any space Y.

Proof. It suffices to show that $f(g_1+g_2) = fg_1 + fg_2$ for all $g_1,g_2 \in DC(Y,X)$. In fact, we have

$$\gamma\tau\{J(fm)\}q(g_1 \times g_2)\Delta = \gamma\Omega(J(fm))e'q(g_1 \times g_2)\Delta ,$$

and hence the assertion follows from Lemma 3.3.6 and the preceding lemma.

Combining Theorem 4.3.2 with Proposition 4.3.4, we have the next result.

Theorem 4.3.5. If $f: X \to A$ is a map of H-groups, then

$$f_{\#}: DC(Y,X) \to DC(Y,A)$$

and

$$f_{\#}: DG(Y,X) \to DG(Y,A)$$

are homomorphisms of abelian groups for any space Y.

In view of the above theorem, we see that for any space Y, $DC(Y,-)$ and $DG(Y,-)$ are covariant functors from the full subcategory of H-groups and maps into the category of abelian groups and homomorphisms.

Example. If $f: K(\pi,n) \to K(\pi,r)$ is a map, then

$$f_{\#}: G^n(Y;\pi) \to G^r(Y;\pi)$$

is a group homomorphism for any space Y.

We are now in a position to state our main theorem.

Theorem 4.3.6. For any H-group X, $DC(X,X)$ and $DG(X,X)$ are rings.

Example 1. For any space X, $DC(\Omega X,\Omega X)$ and $DG(\Omega X,\Omega X)$ are rings.

Example 2. For any Eilenberg-MacLane complex $K = K(\pi,n)$, $DC(K,K)$ and $DG(K,K)$ are rings.

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APPENDIX

5.1 Introduction

In this appendix we shall provide the reader with some discussion and examples for some of the terminology used and results established. Most of these examples were not mentioned explicitly in the previous chapters.

5.2 H-groups

All topological groups are H-groups, since the definition of a topological group is stronger than that of an H-group in the sense that the homotopy relation is replaced by equality.

Example 5.2.1. The loop space ΩX is an H-group for each space X. In fact, for $\ell_1, \ell_2 \in \Omega X$ and $t \in I$, we may define two maps $m: \Omega X \times \Omega X \to \Omega X$ and $\mu: \Omega X \to \Omega X$ by

$$m(\ell_1, \ell_2)(t) = \begin{cases} \ell_1(2t), & 0 \leq t \leq \frac{1}{2} \\ \ell_2(2t-1), \quad \text{if} & \frac{1}{2} \leq t \leq 1 , \end{cases}$$

and $\mu(\ell)(t) = \ell(1-t)$. Then $(\Omega X, m, \mu)$ becomes an H-group.

5.3 H-spaces

All H-groups are H-spaces, since an H-group is an H-space which satisfies some additional conditions (see P4).

Example 5.3.1. S^1, S^3 and S^7 are H-spaces. If we regard S^1, S^3 and S^7 as the sets of complex numbers, quaternions and Cayley numbers respectively of norm one, then they become H-spaces with the map m given respectively by the complex multiplication, the quaternion multiplication and the Cayley multiplication. In fact, S^1 and S^3 are topological groups (S^1 is commutative).

Example 5.3.2. Let X^X be the space of free maps from X into X with the compact-open topology. Let 1_X be the base point of X^X. Let $m: X^X \times X^X \to X^X$ be the composition of maps in X^X. Then (X^X, m) is an H-space. In fact, $m(f, 1_X) = f \circ 1_X = f$ and $m(1_X, g) = 1_X \circ g = g$ for all $f, g \in X^X$. However, (X^X, m) need not be an H-group. For example, let $X = S^2$, since S^2 is not contractible, the constant map $*: S^2 \to S^2$ is not homotopic to the identity map of S^2. Let $g: S^2 \to S^2$ be any map and $f = *: S^2 \to S^2$. Then $m(f,g) = f \circ g = * \neq 1_{S^2}$. This implies that f has no homotopy inverse with respect to m. Hence $(S^{2^{S^2}}, m)$ is not an H-group.

5.4 H-cogroups and co-H-spaces

Example 5.4.1. S^2 is an H-cogroup. To see this, let ϕ be the map of S^2 onto $S^2 \vee S^2$ given by collapsing the equator of S^2 into the base point of $S^2 \vee S^2$ and sending the northern hemisphere and the southern hemisphere of S^2 onto the first copy and the second copy of $S^2 \vee S^2$ respectively. Let $\nu: S^2 \vee S^2$ be the map given by $\nu(x,y,z) = (x,y,-z)$. Then (S^2, ϕ, ν) becomes an H-cogroup. Similarly, any sphere S^n $(n \geq 1)$ is an H-cogroup.

In general, we have

Example 5.4.2. Any suspension ΣX is an H-cogroup. Indeed, for all $x \in X$, $t \in I$, we may let

$$\phi(x,t) = \begin{cases} ((x,2t),*), & \text{if } 0 \leq t \leq \frac{1}{2} \\ (*,(x,2t-1)), & \text{if } \frac{1}{2} \leq t \leq 1 , \end{cases}$$

and $\nu(x,t) = (x,1-t)$. Then $(\Sigma X,\phi,\nu)$ is an H-cogroup.

Remark. The suspension is the only H-cogroup ([17], P4) that I have been able to find.

All H-cogroups are co-H-spaces, since an H-cogroup is a co-H-space which satisfies some additional conditions (see P5). Thus all spheres are co-H-spaces.

5.5 Cyclic maps

We shall now recall how the term "cyclic map" originated. Let X be a topological space. A cyclic homotopy ([9] P840 and [8]) $H: X \times I \to X$ is a homotopy such that $H(x,0) = H(x,1) = x$ for all $x \in X$. If H is a cyclic homotopy, then the path $\sigma: I \to X$ given by $\sigma(t) = H(*,t)$ for all $t \in I$, is called the trace of H. Gottlieb ([9], P840) introduced the subgroup $G(X)$ (of the fundamental group) which consists of all the homotopy classes of those loops which are the trace of some cyclic homotopy. Later, Varadarajan ([31], P141) generalized $G(X)$ to $G(A,X)$ and called the maps $f: A \to X$ which are represented by elements of $G(A,X)$ "cyclic". This is how the

term "cyclic map" first appeared and we stick to it by following Varadarajan's convention.

It was seen in Chapter I that a necessary and sufficient condition for a topological space to be an H-space is that the identity map of X be cyclic. We also saw that if the Gottlieb set $G(A,X)$ vanishes then certain fibrations admit a cross section. Apart from these, some relationships also exist between maps of finite order and cyclic maps and $G(A,-)$ preserves products in the second variable. Thus the theory of cyclic maps and evaluation subgroups could be very interesting and is worth studying.

Although the following example is well-known, we still give it as an illustration.

Example 5.5.1. Let (X,m) be an H-space and A any space. Then any map $f\colon A \to X$ is cyclic. In fact, let $F = m(1\times f)$, $j\colon X \vee A \to X \times A$ and $j'\colon X \vee X \to X \times X$ be the usual inclusions. Then $Fj = m(1\times f)j = mj'(1\vee f) = \nabla(1\vee f)$. Hence f is cyclic.

Example 5.5.2. (see Proposition 1.3.3) Let $g\colon S^8 \to S^5$ be any map. Then $2g$ is cyclic. In fact, S^8 is 9-coconnected and the assertion follows by Proposition 1.3.3.

The next example of cyclic maps is derived from an example of a topological group which is not a Lie group taken from the book: Introduction to topological groups (P82) written by T. Husain.

Example 5.5.3. Let $G = \prod_{i=1}^{\infty} R_i$, the direct product of

countably infinite number of copies of the real line endowed with the product topology (we add elements of G componentwise). Then G is a topological group which is not a Lie group. Let $H = R_1 \times \{0\} \times \{0\} \times \dots$ be the first copy in the product $\prod_{i=1}^{\infty} R_i$. Then H is a closed subgroup of G. Form the coset space G/H. Then $G/H \equiv \{\langle g_i \rangle + H : \langle g_i \rangle \in G\} = \{R_1 \times \{g_2\} \times \{g_3\} \times \dots : \langle g_i \rangle \in G\} \equiv \{R_1 \times \prod_{j=2}^{\infty} \{g_j\} : \langle g_i \rangle \in G\}$. Let $p: G \to G/H$ be the natural map. We claim that p is cyclic.

To see this, let $F: G/H \times G \to G/H$ be the map given by $F(g+H, g') = (g+g') + H$ for all $g, g' \in G$. Then $F(H, g') = (0+g') + H = g' + H = p(g')$ and $F(g+H, 0) = g + H$. Hence p is cyclic.

5.6 Evaluation subgroups

The theory of evaluation subgroups is relatively very new as compared with that of homotopy groups. Although we are able to show that $G(A,X)$ forms a group or even a ring under certain conditions, we make no claim that it be always computable at this time. However, we hope to investigate the problem further in future. This problem is similar to the following situation. As is well-known $[X,Y]$ is always a group if either X is an H-cogroup or Y is an H-group ([17], PP1-4), but in either case the group $[X,Y]$ is not always easy to compute. In fact, X is an H-space iff $G(A,X) = [A,X]$ for any space A.

In what follows, we shall provide a few examples of evaluation subgroups. For more details of the topic the reader is referred to [9], [12] and [23] (the evaluation subgroups of various

Stiefel manifolds were calculated in [23]).

Example 5.6.1. Let K be the Klein bottle. Then $G_1(K) = Z(\pi_1(K))$, the center of $\pi_1(K)$ where $\pi_1(K)$ is the group on two generators a and b with a relation $bab = a$. In fact, using the notion of 2-extensibility, Gottlieb showed ([9], P848) that $G_1(X) = Z(\pi_1(X))$ if X is aspherical, i.e. $\pi_n(X) = 0$ for $n > 1$. Since the Klein bottle is aspherical, the assertion follows.

The following important example is due to Gottlieb and the proof is due to Varadarajan.

Example 5.6.2. For any integer $n \geq 1$,

$$G(S^n,S^n) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 2Z \subset Z = \pi_n(S^n) & \text{if } n \text{ is odd and } n \neq 1,3,7 \\ Z = \pi_n(S^n), & \text{if } n = 1,3,7 . \end{cases}$$

Proof. First observe that $f\colon S^n \to S^n$ is a cyclic map iff there exists a map $F\colon S^n \times S^n \to S^n$ of bidegree (1, deg f).

Case 1. n is even If $F\colon S^n \times S^n \to S^n$ is of bidegree (1, deg f) , applying the Hopf-construction to F we get a map $J(F)\colon S^{2n+1} \to S^{n+1}$ whose Hopf-invariant is equal to deg f . However, it is known that when $n + 1$ is odd the Hopf-invariant of any map $S^{2n+1} \to S^{n+1}$ is zero. Hence the only cyclic maps $S^n \to S^n$ are those of degree zero. Thus $G(S^n,S^n) = 0$.

Case 2. n is odd Let $\eta \in \pi_n(S^n)$ denote a generator. It is known that there exists a map $F\colon S^n \times S^n \to S^n$ of bidegree (1,2) . It follows that the elements

2η is in $G(S^n,S^n)$. It is also known that there exists a map $F: S^n \times S^n \to S^n$ of bidegree $(1,1)$ iff $n = 1, 3$ or 7. Thus the element η itself lies in $G(S^n,S^n)$ iff $n = 1, 3$ or 7. Since $G(S^n,S^n)$ is a subgroup of $\pi_n(S^n) = Z$, the assertion follows. In fact, if $n = 1, 3$ or 7 then S^n is an H-space and $G(S^n,S^n) = \pi_n(S^n) = Z$, from another point of view.

Using a result of Gottlieb, we can obtain the following interesting example.

Example 5.6.3. $G_3(S^2) \cong Z$. To see this, let G be the group of homeomorphisms of S^2 onto itself and $\omega: {S^2}^{S^2} \to S^2$ be the evaluation map. Then $\omega_{\#}\pi_n(G) = \pi_n(S^2)$, $n > 2$ ([9], P856). Thus $G_3(S^2) = \omega_{\#}\pi_3({S^2}^{S^2}, 1_{S^2}) \supset \omega_{\#}\pi_3(G) = \pi_3(S^2)$, so that $G_3(S^2) = \pi_3(S^2) \cong Z$.

5.7 Central maps

The notion of central maps was first introduced by M. Arkowitz and C.R. Curjel in their paper [3]. In Chapter II, we applied this idea by introducing a set $C(A,X)$ to obtain one of our main results, namely, $G(X,X)$ forms a ring for any H-cogroup X. Thus central maps should also be looked at in the study of cyclic maps.

Example 5.7.1. Any map $f: S^1 \to S^1$ is central. In fact let p_1 and p_2 be projections of $S^1 \times S^1$ onto the first and second coordinate spaces respectively. Then $p_1 + p_2 = p_2 + p_1$, since S^1 is a commutative topological group. Thus

$(p_1+p_2) - p_1 - p_2 = 0$. Let m and μ be the multiplication and inversion of S^1 respectively. Then $m = p_1 + p_2$ and $m(\mu\times\mu) = -p_1 - p_2$. It follows that $m + m(\mu\times\mu) = 0$, i.e. $m(m\times m(\mu\times\mu))\Delta_{S^1\times S^1} = 0$. Hence $m(m\times m)(1\times 1\times\mu\times\mu)\Delta_{S^1\times S^1} = 0$, or $c = 0$ and hence $c(1\times f) = 0$ for all $f\colon S^1 \to S^1$. Thus f is central.

Example 5.7.2. Let A be any space and $f\colon A \to \mathbb{R}P^\infty$ (real projective space) a map. Then Ωf is central. In fact, $\mathbb{R}P^\infty$ is a topological group and hence an H-space. By Example 5.5.1, there exists a map $F\colon \mathbb{R}P^\infty \times A \to \mathbb{R}P^\infty$ such that $Fj = \nabla(1\vee f)$ where j is the obvious inclusion. Thus $\nabla(1\vee f)L = FjL \simeq *$ where $L\colon \mathbb{R}P^\infty \flat A \to \mathbb{R}P^\infty \vee A$ is the usual inclusion. It follows that Ωf is central by Lemma 1.3.9.

5.8 Cocyclic maps and concentral maps

The following are two elementary examples of cocyclic maps.

Example 5.8.1. Every constant map $*\colon X \to A$ is cocyclic. In fact, we may take the inclusion map $i_1\colon X \to X \vee A$ to be a coassociated map of $*$.

Example 5.8.2. If X is a co-H-space, then every map $f\colon X \to A$ is cocyclic. To see this, let $\psi\colon X \to X \vee X$ be the co-H-structure on X . Then we may take $(1\vee f)\psi$ to be a coassociated map of f .

The next example is not quite trivial and reference of [17] is required.

Example 5.8.3. Consider the cofibration $A \vee B \xrightarrow{j} A \times B \to A \wedge B \equiv \dfrac{A \times B}{A \vee B}$ where A and B are spaces of the homotopy type of CW complexes. Then the coboundary map ∂ in the Puppe sequence of the cofibration is cocyclic. To see this, we first describe how ∂ is obtained. For the sake of simplicity, let $X = A \vee B$, $Y = A \times B$ and $Z = A \wedge B$. Let $W \equiv Y \cup_j CX$ be the mapping cone of $j\colon X \to Y$. It is well-known that the map $q\colon W \to Y/X$ which collapses CX to the base point is a homotopy equivalence. Let h be a homotopy inverse of g. If we collapse Y to the base point in W, we get ΣX. Let $i\colon W \to \Sigma X$ be the identification map. Let $\partial = ih\colon Y/X \to \Sigma X$. Then ∂ is the coboundary map in the following Puppe sequence of the given cofibration:

$$X \xrightarrow{j} Y \to Z \xrightarrow{\partial} \Sigma X \to \Sigma Y \to \quad \ldots \quad .$$

We shall now show that ∂ is cocyclic. According to [17] (P171), there exists a cooperation $\phi\colon Z \to Z \vee \Sigma X$ with $p_1\phi \simeq 1_Z$ where $p_1\colon Z \vee \Sigma X \to Z$ is projection, and $\partial^{\#} = s^{\#}$ ([17], P176) where $s = p_2\phi$ and $p_2\colon Z \vee \Sigma X \to \Sigma X$ is projection. Thus $\partial^{\#}[1_{\Sigma X}] = s^{\#}[1_{\Sigma X}]$ i.e. $1_{\Sigma X}\circ\partial \simeq 1_{\Sigma X}\circ s$ or $\partial \simeq s$. Claim: s is cocyclic. Indeed, we have

$$\begin{aligned}(1\times s)\Delta &= (1\times p_2\phi)\Delta \simeq (p_1\phi\times p_2\phi)\Delta \\ &= (p_1\times p_2)(\phi\times\phi)\Delta = (p_1\times p_2)\Delta\phi \\ &= \phi = j\phi\end{aligned}$$

where j is the obvious inclusion. Thus s and hence ∂ is cocyclic.

Note that Example 5.8.1 is also an example of cocentral

maps. Another example is the following.

Example 5.8.4. If G is a homotopy commutative H-cogroup (e.g. $\Sigma^2 X$ for any space X) then any map $f\colon G \to G$ is cocentral (compare Example 5.7.1).

5.9 Examples on some results

The following example shows that the hypothesis on X in Corollary 1.4.6 is necessary.

Example 5.9.1. Let $X = A = S^7$ in the Corollary. Since $G(S^7,S^7) = \pi_7(S^7) = Z$, every map $f\colon S^7 \to S^7$ is cyclic. However, $\Sigma f\colon S^8 \to S^8$ is not of finite order unless $\Sigma f \simeq *$, since $\Sigma f \in \pi_8(S^8) = Z$. Observe that $X = S^7$ does not satisfy the hypothesis of Corollary 1.4.6.

Example 5.9.2 (Corollary 2.2.5) Let X = the torus, $Y = S^2$ and $A = S^3$. Then we have

$$\begin{aligned} G(A,X\times Y) &= G(S^3,X\times S^2) \\ &\cong G(S^3,X) \oplus G(S^3,S^2) \\ &\cong G(S^3,S^1\times S^1) \oplus Z \text{ , by Example 5.6.3} \\ &\cong G(S^3,S^1) \oplus G(S^3,S^1) \oplus Z \\ &\cong Z \text{ , since } G(S^3,S^1) = 0 \text{ .} \end{aligned}$$

Example 5.9.3 (Theorem 2.3.2) Let $A = S^3$ and $X = S^2$. Then $G(S^3,S^2) = G(S^3,\Sigma S^1) = C(S^3,\Sigma S^1)$, by Theorem 2.3.2. Thus $C(S^3,S^2) = C(S^3,\Sigma S^1) = G(S^3,S^2) = Z$, by Example 5.6.3. Also

$G(S^3,\Sigma S^2) = G(S^3,S^3) = Z$, and $C(S^3,\Sigma S^2) = G(S^3,\Sigma S^2)$ by Theorem 2.3.2. Hence $C(S^3,S^3) = G(S^3,S^3) = Z$.

Remark. It is not known to us whether there exist spaces A and X for which $G(A,X) \subsetneq C(A,X)$.

Example 5.9.4 (Theorem 2.3.8) Since all spheres S^n $(n \geq 1)$ are suspensions, we have $G(S^n,S^n) = C(S^n,S^n) = W(S^n,S^n) = P(S^n,S^n)$ by Theorem 2.3.8. By Example 5.6.2, this common value is given by

$$G(S^n,S^n) = \begin{cases} 0\ , & \text{if } n \text{ is even} \\ 2Z, & \text{if } n \text{ is odd and } n \neq 1,3,7 \\ Z\ , & \text{if } n = 1,\ 3 \text{ or } 7\ . \end{cases}$$